XIII. *On the Dynamics of a Rigid Body in Elliptic Space.*

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THIS paper is an attempt to work out the theory of the motion of a rigid body under the action of any forces, with the generalised conceptions of distance of the so-called non-Euclidean geometry. Of the three kinds of non-Euclidean space, that known as elliptic space has been chosen, because of the perfect duality and symmetry which exist in this case. The special features of the method employed are the extensive use of the symmetrical and homogeneous system of coordinates given by a quadrantal tetrahedron, and the use of Professor CAYLEY'S coordinates, in preference to the "rotors" of Professor CLIFFORD, to represent the position of a line in space.

The first part, §§ 1–21, is introductory; in it the theory of plane and solid geometry is briefly worked out from the basis of Professor CAYLEY'S idea of an absolute quadric. By taking a quadrantal triangle (*i.e.*, a triangle self-conjugate with regard to the absolute conic) as the triangle of reference, the equations to lines, circles, and conics are found in a simple form, and some of their properties investigated.

The geometry of any plane is proved to be the same as that of a sphere of unit radius, so that elliptic space is shown to have a uniform positive curvature.

The theory is then extended to solid geometry, and the most important relations of planes and lines to each other are worked out.

The next part treats of the kinematics of a rigid body. The possibility of the existence of a rigid body is shown to be implied by the constant curvature of elliptic space, and then the theory of its displacement is made to depend entirely on orthogonal transformation. Any displacement may be expressed as a twist about a certain screw. A rotation about a line is shown to be the same as an equal translation along its polar; so that the difference between a rotation and a translation disappears, and the motion of any body is expressed in terms of six symmetrical angular velocities. An angular velocity ω, about a line whose coordinates are a, b, c, f, g, h, is found to be capable of resolution into component angular velocities, $a\omega, b\omega \ldots h\omega$, about the edges of the fundamental tetrahedron.

The theory of screws is next considered. A twist on a screw can be replaced by a pair of rotations about any two lines which are conjugate to each other in a certain linear complex. The surface corresponding to the cylindroid is found to be of the fourth order with a pair of nodal lines. Lastly, the condition of equivalence of any number of twists about given screws is investigated.

In kinetics, the measure of force is deduced from NEWTON'S second law of motion, and the laws of combination and resolution are proved. The consideration of the whole momentum of a body suggests the idea of moments of inertia, and a few of their properties are investigated. The general equations of motion referred to any moving axes are then found, and in a particular case they reduce to a form corresponding to EULER'S equations; these are of the type

$$A\dot{\omega}_1-(B-H)\omega_2\omega_6-(G-C)\omega_5\omega_3=Q_1.$$

The last part is occupied in the solution of these equations when no forces act, in terms of the theta-functions of two variables. A solution is obtained in the form

$$\omega_1=a.\frac{\vartheta_0(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_4=f.\frac{\vartheta_7(x, y)}{\vartheta_{12}(x, y)},$$

$$\omega_2=b.\frac{\vartheta_2(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_5=g.\frac{\vartheta_5(x, y)}{\vartheta_{12}(x, y)},$$

$$\omega_3=c.\frac{\vartheta_9(x, y)}{\vartheta_{12}(x, y)}, \qquad \omega_6=h.\frac{\vartheta_{14}(x, y)}{\vartheta_{12}(x, y)},$$

where $x=nt+\alpha$ and y is arbitrary. But in order that these values may satisfy the equations, a relation among the parameters of the theta-functions must be satisfied. This is

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0.$$

The solution is not complete, because after satisfying the equations of motion only four constants remain to express the initial conditions, whereas six constants are required.

Introduction.

A concise review of the characteristics of the different kinds of generalised space will be found in the introduction to Professor CLIFFORD'S mathematical works by the late Professor H. J. S. SMITH (Introduction, p. XXXIX.), together with an analysis of CLIFFORD'S numerous memoirs relating to this subject. Further information may be found in the following papers :—

Dr. BALL, "On the Non-Euclidean Geometry," 'Hermathena,' vol. iii.

Professor CAYLEY, "A Sixth Memoir on Quantics," Phil. Trans., 1859.

Professor LINDEMANN, "Projectivische Behandlung der Mechanik starrer Körper," Math. Annalen, Bd. vii., 1874.

Mr. HOMERSHAM COX, "Homogeneous Coordinates in Imaginary Geometry, and their application to Systems of Forces," Quarterly Journal, vol. 18.

For the coordinates of a line see the paper by Professor CAYLEY, Camb. Phil. Trans., vol. xi.

On the geometry of elliptic space.

§ 1. Geometrical theorems are sometimes divided into two classes, descriptive and metrical. Descriptive theorems have reference to the relative positions of figures, and are unaltered by projection and linear transformation. Metrical theorems have reference to magnitudes, such as lengths of lines, the measures of angles, areas and volumes. But it has been pointed out by Professor CAYLEY that metrical theorems may always be stated as descriptive; they are descriptive relations between geometrical figures and certain fixed geometrical forms, which he calls the Absolute. In ordinary plane geometry the Absolute consists of an imaginary point-pair on a real line, viz., the circular points at infinity. The magnitude of the angle between two lines, for instance, may be expressed as a function of the anharmonic ratio of the pencil formed by the lines, and the pair of lines drawn from their intersection to the Absolute point-pair. In three dimensions the Absolute is the imaginary circle at infinity.

2. Professor CAYLEY generalises this idea of metrical theorems by supposing the Absolute to be the points and planes of a fixed quadric surface in space. The Absolute in any plane consists of the points and lines of a fixed conic lying in the plane, the conic being the intersection of the plane with the Absolute quadric.

There are three different kinds of geometry of space depending on the nature of this Absolute quadric. These are

(1.) Elliptic geometry, in which all the elements of the Absolute are imaginary.

(2.) Hyperbolic geometry, in which the Absolute surface is real, but contains no real straight lines, and surrounds us.

(3.) Parabolic geometry, in which the Absolute degenerates into an imaginary conic in a real plane.

In what follows we shall suppose all the elements of the Absolute imaginary.

3. On any line there is an Absolute point-pair, viz., the intersections of the line with the Absolute quadric. The position of any point on the line will be determined when we know the ratio of its distances from the Absolute points. If we denote this ratio by z, the distance between two points must be a function of the ratios z_1 and z_2, corresponding to the points.

Now, the fundamental property of the distance between two points may be expressed by the relation

$$\overline{PQ}+\overline{QR}=\overline{PR}$$

where P, Q, R are three points on the same line. In view of this relation the distance between two points z_1, z_2 is defined to be

$$c \log \frac{z_1}{z_2}$$

where c is an arbitrary constant. Hence, in this generalised system of Geometry, the distance between two points on a line is measured by the logarithm of the anharmonic ratio of the range formed by the two points and the absolute point-pair of the line, multiplied by an arbitrary constant.

4. The relations between lines passing through a point and lying in a plane are exactly the same as the relations between points along a line. Among the lines lying in a plane and passing through a point there are two fixed lines called the Absolute pair of lines; these are the pair of tangents that can be drawn from the point to the Absolute conic of the plane. The measurement of angles will thus be exactly similar to the measurement of distances. The angle between two lines lying in a plane is measured by the logarithm of the anharmonic ratio of the pencil formed by the lines and the Absolute pair of lines passing through the point, multiplied by an arbitrary constant. There is a special advantage in choosing both these arbitrary constants to be $\frac{i}{2}$, where i denotes $\sqrt{-1}$.

From these definitions it follows by properties of poles and polars that the distance between two points is equal to the angle between their polars, so that any theorem of distances has a reciprocal theorem relating to angles.

5. Let $U=0$ be the equation to the Absolute conic in any plane in the notation of Ordinary Geometry. If (x_1, y_1, z_1), (x_2, y_2, z_2) be any two points, the coordinates of any point on the line joining them are proportional to $x_1-\lambda x_2$, $y_1-\lambda y_2$, $z_1-\lambda z_2$. Hence to find the Absolute point-pair we have the equation

$$U_{11}-2\lambda U_{12}+\lambda^2 U_{22}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with the usual notation.

Let δ be the generalised distance between the points 1, 2. Then

$$\delta=\frac{i}{2}\log\frac{\lambda_1}{\lambda_2}$$

where λ_1, λ_2 are the roots of the quadratic equation (1). Hence

$$\epsilon^{2\delta i}+\epsilon^{-2\delta i}=\frac{\lambda_1{}^2+\lambda_2{}^2}{\lambda_1\lambda_2}$$

therefore

$$\left\{\frac{\epsilon^{\delta i}+\epsilon^{-\delta i}}{2}\right\}^2=\frac{(\lambda_1+\lambda_2)^2}{4\lambda_1\lambda_2}$$

that is,

$$\cos^2\delta=\frac{U_{12}}{U_{11}.U_{22}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

6. As the triangle of reference take any self-conjugate triangle with respect to the Absolute conic, so that the equation to the conic becomes

$$px^2+qy^2+rz^2=0.$$

Then $U_{12}=px_1x_2+qy_1y_2+rz_1z_2$, and

$$\cos^2\delta=\frac{\{px_1x_2+qy_1y_2+rz_1z_2\}^2}{(px_1^2+qy_1^2+rz_1^2)(px_2^2+qy_2^2+rz_2^2)}$$

This suggests a new system of coordinates. Let (x, y, z) denote the cosines of the generalised distances from the angular points of the triangle of reference, of a point whose coordinates were (x_1, y_1, z_1) in Ordinary Geometry.

Then

$$x^2=\frac{px_1^2}{px_1^2+qy_1^2+rz_1^2}, \text{ \&c.}$$

Hence

$$x^2+y^2+z^2=1,$$

and the equation to the Absolute conic is

$$x^2+y^2+z^2=0.$$

Then if δ denote the distance between two points (x, y, z), (x', y', z'),

$$\cos\delta=xx'+yy'+zz'.$$

7. If (l, m, n) be the coordinates of any point, the equation to its polar line with respect to the Absolute conic is

$$lx+my+nz=0.$$

Here (l, m, n) may be looked upon as the coordinates of the pole, or the tangential coordinates of the line, indifferently; and we shall always suppose that

$$l^2+m^2+n^2=1.$$

The form of the equation shows that a point is distant one right angle from any point of its polar. From this theorem, we deduce by Reciprocation, that a given line is perpendicular to any line through its pole. Hence the sides and angles of any self-conjugate triangle with respect to the Absolute conic are all right angles. Such a triangle is called a Quadrantal triangle. We can now give a new interpretation of the coordinates (x, y, z); they are the sines of the perpendiculars from the point let fall on the three sides of the triangle of reference.

8. The angle between two lines is equal to the distance between their poles; *i.e.*, if θ be the angle between two lines whose poles are (l, m, n), (l', m', n'),

$$\cos\theta = ll' + mm' + nn'.$$

To draw a perpendicular from a point to a line, we have only to join the point to the pole of the line. In general we can draw only one perpendicular from the point to the line, but if the point be the pole of the line, every line through it is a perpendicular to the given line. Let ϖ be the sine of the perpendicular from a point (x, y, z) to a line

$$lx + my + nz = 0.$$

Then ϖ denotes the cosine of the distance of the point from the pole of the line, therefore

$$\varpi = lx + my + nz.$$

The equation to the line joining two points (x', y', z'), (x'', y'', z'') is

$$\begin{vmatrix} x & y & z \\ x' & y' & z \\ x'' & y'' & z'' \end{vmatrix} = 0$$

The equation to a line drawn from a point (x', y', z') perpendicular to a line $lx + my + nz = 0$ is

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ l & m & n \end{vmatrix} = 0$$

9. We now proceed to establish the Trigonometry of any plane. Let A B C be a triangle, and, for simplicity, let C be an angle of the triangle of reference, and let A, B be the points 1, 2. Denoting the sides of this triangle by a, b, c, we get

$$\cos c = x_1x_2 + y_1y_2 + z_1z_2$$
$$\cos a = z_2$$
$$\cos b = z_1$$

Again, the equation to the absolute pair of lines through C is

$$x^2 + y^2 = 0.$$

It is easy to see that the formula

$$\cos^2\delta = \frac{U_{12}^2}{U_{11}U_{22}}$$

is applicable to this case also, therefore

$$\cos^2 C = \frac{(x_1x_2+y_1y_2)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}.$$

and therefore

$$x_1x_2+y_1y_2=\sin a \sin b \cos C$$

Hence finally

$$\cos c=\cos a \cos b+\sin a \sin b \cos C.$$

From this equation may be deduced, as in Spherical Trigonometry, the relations

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}.$$

Hence the geometry of any plane in elliptic space is the same as the geometry of a sphere in ordinary space. A straight line in the plane corresponds to a great circle on the sphere. But further, the distance between any two points measured in the way we have indicated is periodic, the length of a complete period being 2π. Hence we infer that the radius of any great circle of the sphere is unity. Thus any line and any plane may be supposed to have a uniform positive curvature unity.

RIEMANN, in his memoir "On the Hypotheses which lie at the Bases of Geometry," speaks of the curvature of an n-fold extent at a given point and in a given surface direction; he explains it as follows :—

Suppose that from any given point the system of shortest lines going out from it be constructed. Any one of these geodesics is entirely determined when its initial direction is given. Accordingly we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface direction; this surface has at the given point a definite curvature, measured in the manner indicated by GAUSS. This curvature is the curvature of the n-fold continuum at the given point in the given surface direction.

If we construct a surface at a given point of elliptic space in any direction in the way thus indicated, the geometry of such a surface is the same as that of a sphere of unit radius in ordinary space. Thus for all points and for all surface directions the curvature will be unity. Hence elliptic space is said to have a uniform positive curvature.

10. The general equation of a conic, in the notation of ordinary space, is a homogeneous equation of the second degree. Hence, when we pass to the new coordinates, the equation to a conic will still be homogeneous and of the second degree. If we choose our triangle of reference to be the self-conjugate triangle common to the conic and the Absolute, the form of the equation becomes

$$Ax^2+By^2+Cz^2=0.$$

The sides of this triangle of reference may be called the principal axes of the conic.

The condition that an equation of the second degree should represent two straight lines, is that the discriminant should vanish. If we consider the equation

$$S=k(x^2+y^2+z^2),$$

and make the discriminant vanish, we get three pairs of lines, which are the representatives of the asymptotes in ordinary geometry. If we take the form of the equation referred to the principal axes, we see that a pair of these lines passes through each angular point of the triangle. The asymptotes, however, no longer *touch* the conic, but are the six lines joining the four points of intersection of the conic with the absolute.

11. If the equation to a conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

the tangential equation becomes

$$a^2l^2+b^2m^2+c^2n^2=0.$$

The tangential equation of the absolute is

$$l^2+m^2+n^2=0.$$

The foci of a conic may be defined to be the six points of intersection of the common tangents to the conic and the Absolute. Confocal conics are those which have the same common tangents with the Absolute. Hence the tangential equation of a system of confocal conics is

$$a^2l^2+b^2m^2+c^2n^2+\lambda(l^2+m^2+n^2)=0$$

and therefore, in point coordinates, the equation to a system of confocal conics is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=0.$$

Thus confocal conics have the same principal axes. Also it may be shown in the usual way that confocal conics cut at right angles, and that two confocals can be drawn through any point in the plane.

12. The equation to a circle whose centre is (l, m, n), and the cosine of whose radius is r, is

$$lx+my+xz=r;$$

or, making it homogeneous,

$$(lx+my+nz)^2=r^2(x^2+y^2+z^2).$$

The form of this equation shows that a circle is a conic having double contact with the Absolute in the two points in which it is met by the polar line of the centre of the circle.

Many other interesting properties of conics may be worked out by means of these equations, but as they will not concern us we pass on at once to the geometry of three dimensions.

Solid geometry.

13. As before, we refer the Absolute quadric to a self-conjugate tetrahedron. Let the equation to it, in any system of plane coordinates of ordinary space, be

$$p\alpha^2+q\beta^2+r\gamma^2+s\delta^2=0.$$

Then, if θ be the distance between the points 1, 2,

$$\cos^2\theta=\frac{[p\alpha_1\alpha_2+q\beta_1\beta_2+r\gamma_1\gamma_2+s\delta_1\delta_2]^2}{[p\alpha_1^2+q\beta_1^2+r\gamma_1^2+s\delta_1^2][p\alpha_2^2+q\beta_2^2+r\gamma_2^2+s\delta_2^2]}.$$

This again suggests a new system of homogeneous coordinates. Let (x, y, z, u) denote the cosines of the distances of the point $(\alpha, \beta, \gamma, \delta)$ from the four angular points of the tetrahedron of reference. Then

$$x^2=\frac{p\alpha^2}{p\alpha^2+q\beta^2+r\gamma^2+s\delta^2}, \text{ \&c.}$$

For any real point

$$x^2+y^2+z^2+u^2=1.$$

The equation of the Absolute quadric in these coordinates is

$$x^2+y^2+z^2+u^2=0.$$

Also if θ be the distance between two points (x, y, z, u) (x', y', z', u'), we have

$$\cos\theta=xx'+yy'+zz'+uu'.$$

14. If (l, m, n, p) be the coordinates of a point the equation of the polar plane with reference to the Absolute quadric is

$$lx+my+nz+pu=0.$$

In the equation to any plane we shall suppose the coefficients such that

$$l^2+m^2+n^2+p^2=1$$

and then (l, m, n, p) will be regarded indifferently as the coordinates of the pole, or the coordinates of the plane. The distance of a point from any point of its polar plane is a right angle; and from what was proved for two dimensions it follows that any line passing through the pole of a plane is perpendicular to the plane. The lengths of the six edges and the angles of all the faces of the fundamental tetrahedron are all right angles. Such a tetrahedron is called a quadrantal tetrahedron. The coordinates (x, y, z, u) are the sines of the perpendicular distances of a point from the four planes of reference. If we put $u=0$ in any formula the system reduces to the same coordinates as were used in two dimensions. Let ϖ denote the sine of the perpendicular from any point (x, y, z, u) to the plane

$$lx+my+nz+pu=0,$$

then ϖ is the cosine of the distance between (x, y, z, u) and the pole of the plane, and therefore

$$\varpi=lx+my+nz+pu.$$

The angle between two planes is equal to the distance between their poles, so that if θ be the angle between the two planes (l, m, n, p), (l', m', n', p'),

$$\cos\theta=ll'+mm'+nn'+pp'.$$

15. A straight line may be conveniently specified by six coordinates, as shown by Professor CAYLEY. Let (x, y, z, u), (x', y', z', u') be two points on any line, and (l, m, n, p), (l', m', n', p') two planes through it, so that

$$\left.\begin{array}{l} lx+my+nz+pu=0 \\ l'x+m'y+n'z+p'u=0 \end{array}\right\}$$

$$\left.\begin{array}{l} lx'+my'+nz'+pu'=0 \\ l'x'+m'y'+n'z'+p'u'=0 \end{array}\right\}$$

Eliminating l between the first and third equations we get

$$0+m(xy'-x'y)-n(zx'-z'x)+p(xu'-x'u)=0$$

Similarly,

$$0+m'(xy'-x'y)-n'(zx'-z'x)+p'(xu'-x'u)=0,$$

and we can obtain other equations of similar forms in the same way. Let a, b, c, f, g, h denote respectively the quantities

$$yz'-y'z,\ zx'-z'x,\ xy'-x'y,\ xu'-x'u,\ yu'-y'u,\ zu'-z'u,$$

then

$$\left.\begin{array}{rrrrl} . & -cm & +bn & -fp & =0 \\ cl & . & -an & -gp & =0 \\ -bl & +an & . & -hp & =0 \\ fl & +gm & +hn & . & =0 \end{array}\right\}$$

There is a similar set of equations obtained by writing l', m', n', p' for l, m, n, p in these equations.

Taking the equations

$$\left.\begin{array}{l} -cm+bn-fp=0 \\ -cm'+bn'-fp'=0 \end{array}\right\}$$

and eliminating f, we have

$$c(mp'-m'p)=b(np'-n'p)$$

Proceeding in this way, and eliminating the other letters in turn, it is easily seen that

$$\begin{array}{cccccc} a & : b & : c & : f & : g & : h \\ =lp'-l'p & : mp'-m'p & : np'-n'p & : mn'-m'n & : nl'-n'l & : l'm-l'm. \end{array}$$

Choose the planes (l, m, n, p), (l', m', n', p') to be at right angles, and the points (x, y, z, u), (x', y', z', u') to be distant a right angle; then

$$\begin{aligned} &(lp'-l'p)^2+(mp'-m'p)^2+(np'-n'p)^2+(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2 \\ &\quad=(l^2+m^2+n^2+p^2)(l'^2+m'^2+n'^2+p'^2)-(ll'+mm'+nn'+pp')^2 \\ &\quad=1, \end{aligned}$$

and similarly it may be shown that

$$(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2+(xp'-x'p)^2+(yp'-y'p)^2+(zp'-z'p)^2=1.$$

Hence

$$a=yz'-y'z=lp'-l'p,$$

and so on for all the letters, and

$$a^2+b^2+c^2+f^2+g^2+h^2=1.$$

From the forms of a, b, c, f, g, h, it is easy to show that there is an identical relation between them

$$af+bg+ch=0.$$

2 P 2

The quantities a, b, c, f, g, h, are thus reduced to four independent variables and are called the six coordinates of the line.

16. If we interchange the letters (x, y, z, u) with the letters (l, m, n, p), we get the polar line of the first. Hence the coordinates of the polar line are

$$f, g, h, a, b, c.$$

The co-ordinates of the line joining two points (x, y, z, u), (x', y', z', u') distant an angle θ from each other, are

$$a=\frac{yz'-y'z}{\sin\theta} \text{ \&c.}$$

Similarly for the line of intersection of two given planes.

It has been incidentally proved that the conditions that a line a, b, c, f, g, h should lie in a plane l, m, n, p, are

$$\left.\begin{array}{llll} . & -cm+bn & -fp & =0 \\ cl & \quad . \quad -an & -gp & =0 \\ -bl+am & \quad . & -hp & =0 \\ fl & +gm+hn & \quad . & =0 \end{array}\right\}$$

which are equivalent to two independent relations. These are also the conditions that the polar line f, g, h, a, b, c should pass through the point (l, m, n, p).

The coordinates of a line through a point (x, y, z, u) perpendicular to a plane (l, m, n, p) are

$$a=\frac{yn-zm}{\sin\theta} \text{ \&c.}$$

where θ is the angle between (x, y, z, u) and the point (l, m, n, p), which is the pole of the plane.

17. We shall now find the length of the perpendicular from any point (x, y, z, u) to a line a, b, c, f, g, h.

Let (l, m, n, p), (l', m', n', p') be two perpendicular planes passing through the line. Let ϖ_1, ϖ_2 be the sines of the perpendiculars from (x, y, z, u) on these planes, and let ϖ be the sine of the perpendicular on the given line. Then by Spherical Trigonometry

$$\varpi^2=\varpi_1^2+\varpi_2^2.$$

Now

$$\left.\begin{array}{l} \varpi_1=lx+my+nz+pu \\ \varpi_2=l'x+m'y+n'z+p'u \end{array}\right\}$$

therefore

$$\left.\begin{aligned}\varpi_1 l' - \varpi_2 l &= \quad . \quad -hy+gz-au\\ \varpi_1 m'-\varpi_2 m &= \quad hx \quad . \quad -fz-bu\\ \varpi_1 n' - \varpi_2 n &= -gx+fz \quad . \quad -cu\\ \varpi_1 p' - \varpi_2 p &= \quad ax+by+cz \quad .\end{aligned}\right\}$$

If we square and add these equations, the left side becomes $\varpi_1^2+\varpi_2^2$; hence

$$\begin{aligned}\varpi^2 = a^2(x^2+u^2)+b^2(y^2+u^2)+c^2(z^2+u^2)+f^2(y^2+z^2)+g^2(z^2+x^2)+h^2(x^2+y^2)\\ +2yz(bc-gh)+2zx(ca-hf)+2xy(ab-fg)\\ +2xu(cg-bh)+2yu(ah-cf)+2zu(bf-ag).\end{aligned}$$

18. Any line which meets a given line and its polar, cuts both perpendicularly, and the length of the part intercepted between them is a right angle. If we have two given lines in space, which do not meet, we can, in general, draw two lines cutting both perpendicularly; these are the two lines which can be drawn meeting the two given lines and their polars. These two common perpendiculars are conjugate to each other with reference to the absolute.

Let (a, b, c, f, g, h), (a', b', c', f', g', h') be two given non-intersecting lines, and let δ be the length of one of their common perpendiculars. Draw a plane (l, m, n, p) through δ and the first line, and another plane (l', m', n', p') through δ and the second line, and let θ be the angle between these planes. Draw also a plane $(\lambda, \mu, \nu, \varpi)$ through the first line and perpendicular to the plane (l, m, n, p), and $(\lambda', \mu', \nu', \varpi')$ through the second line and perpendicular to the plane (l', m', n', p'). Then we have the following relations:—

$$\left.\begin{aligned}l\lambda+m\mu+n\nu+p\varpi&=0\\ l\lambda'+m\mu'+n\nu'+p\varpi'&=0\\ l'\lambda+m'\mu+n'\nu+p'\varpi&=0\\ l'\lambda'+m'\mu'+n'\nu'+p'\varpi'&=0\end{aligned}\right\}$$

$$\left.\begin{aligned}ll'+mm'+nn'+pp'&=\cos\theta\\ \lambda\lambda'+\mu\mu'+\nu\nu'+\varpi\varpi'&=\cos\delta\end{aligned}\right\}$$

Now

$$a=m\nu-\mu n, \ \&c.$$
$$a'=m'\nu'-\mu'n', \ \&c.$$

Hence it follows that

$$\begin{aligned}&aa'+bb'+cc'+ff'+gg'+hh'\\ &=(ll'+mm'+nn'+pp')(\lambda\lambda'+\mu\mu'+\nu\nu'+\varpi\varpi')\\ &\quad-(l\lambda'+m\mu'+n\nu'+p\varpi')(l'\lambda+m'\mu+n'\nu+p'\varpi),\end{aligned}$$

and therefore

$$\cos\theta\cos\delta = aa' + bb' + cc' + ff' + gg' + hh'.$$

19. Again, if we expand the determinant

$$\begin{vmatrix} l & m & n & p \\ \lambda & \mu & \nu & \varpi \\ l' & m' & n' & p' \\ \lambda' & \mu' & \nu' & \varpi' \end{vmatrix}$$

it becomes

$$af' + bg' + ch' + fa' + gb' + hc'.$$

Now, squaring the determinant, we get

$$\Delta^2 = \begin{vmatrix} 1 & . & \cos\theta & . \\ . & 1 & . & \cos\delta \\ \cos\theta & . & 1 & . \\ . & \cos\delta & . & 1 \end{vmatrix}$$

$$= 1 - \cos^2\theta - \cos^2\delta + \cos^2\theta\cos^2\delta$$

$$= \sin^2\theta\sin^2\delta.$$

Hence

$$\sin\theta\sin\delta = af' + bg' + ch' + fa' + gb' + hc'.$$

These formulæ remain unchanged if we pass to the other common perpendicular, or if we take the two polar lines instead of the given lines.

If the lines meet one another

$$af' + bg' + ch' + fa' + gb' + hc' = 0,$$

and then the angle between them is given by the equation

$$\cos\theta = aa' + bb' + cc' + ff' + gg' + hh'.$$

20. The equation to a sphere, whose centre is (l, m, n, p) and the cosine of whose radius is r, is

$$lx + my + nz + pu = r.$$

This may be written in the homogeneous form

$$\{lx + my + nz + pu\}^2 = r^2(x^2 + y^2 + z^2 + u^2).$$

Hence we infer that a sphere is a quadric touching the absolute quadric along its intersection with the polar plane of the centre of the sphere. The polar plane itself is a particular case of a sphere, the radius being equal to a right angle.

21. The general equation of a quadric is a homogeneous equation of the second degree in x, y, z, u. By an orthogonal transformation we know that we can rid the equation of the products of (x, y, z, u), and at the same time keep $x^2+y^2+z^2+u^2$ unchanged. The equation will then reduce to the form

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+\mathrm{D}u^2=0.$$

The tetrahedron of reference is the common self-conjugate tetrahedron to the given quadric and the absolute. The six edges may be called the principal axes of the quadric.

The equation to a system of confocal quadrics may, as before, be shown to be of the form

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}+\frac{u^2}{d^2+\lambda}=0.$$

Hence confocal quadrics cut at right angles.

The kinematics of a rigid body.

22. By a rigid body we mean a collection of particles so bound together that the distance between any pair of them remains the same, however the system be moved in space. In general, if we assume an arbitrary system of measure-relations as the basis of our definition of distance, a rigid body could not exist. But it is pointed out by RIEMANN in his paper " On the Hypotheses which lie at the Bases of Geometry," that the special character of those centinua, whose curvature is constant, is that figures may be moved in them from one position to another without stretching. This may be illustrated for two dimensions, by saying that any figure traced on a spherical surface may be moved from one position to another on the surface without deformation. But on the other hand, a figure traced on the surface of an ellipsoid, or other surface for which the curvature is not uniform, can exist in one position only. Now in elliptic space there is a uniform positive curvature; hence we assume that a figure which exists in one position can exist in any other position of space without changing the distance between any two of its points. The same result is arrived at by KLEIN, by showing the possibility of finding a linear transformation, which transforms the absolute quadric into itself.

23. A point is always distant a right angle from any point of its polar plane. Hence a point and its polar plane always move together like a rigid body.

A displacement which leaves all the points of a given line unchanged in position, is called a rotation about that line. If we take any two fixed points on the line, the distance of any point of the body from each of these remains unchanged. Hence it follows, by Spherical Trigonometry, that the point describes a circle, whose centre is the foot of the perpendicular let fall from the point to the axis of rotation. In the case in which the point lies on the polar line of the axis of rotation, the radius of the circle is a right angle, or in other words the circle becomes a straight line, viz., the polar line itself; so that any point on the polar line of the axis of rotation remains on that polar line.

A displacement which leaves all the planes through a given line unchanged in position is called a translation along that line. In such a displacement the poles of these fixed planes will be fixed points; in other words, all the points of the polar line are fixed. Hence a translation along any line is also a rotation about the polar line. If we measure a translation by the distance through which any point of the line of translation is moved, and a rotation by the angle through which any plane through the axis of rotation is turned, we see that a translation along any line is exactly the same thing as an equal rotation about the polar line.

In elliptic space a translation has a definite line associated with it, just in the same way that a rotation has a definite axis. A translation through four right angles brings a body back to its original position.

24. In working out the kinematics of a rigid body, we shall suppose a quadrantal tetrahedron fixed in the body, so that the whole theory will depend on orthogonal transformation. Let $(l, m, n, p)_{1, 2, 3, 4}$ be the coordinates of the four angular points of the quadrantal tetrahedron moving with the body, referred to a fixed quadrantal tetrahedron in space. Let (x, y, z, u) be the coordinates of a point referred to the fixed tetrahedron, (x_0, y_0, z_0, u_0) the coordinates of the same point referred to the other tetrahedron.

Then

$$\left.\begin{aligned} x_0&=l_1x+m_1y+n_1z+p_1u \\ y_0&=l_2x+m_2y+n_2z+p_2u \\ z_0&=l_3x+m_3y+n_3z+p_3u \\ u_0&=l_4x+m_4y+n_4z+p_4u \end{aligned}\right\}$$

If we square these equations and add them and make

$$x_0^2+y_0^2+z_0^2+u_0^2=1$$

for all values of (x, y, z, u), we find relations among the coefficients, of the types

$$\left.\begin{aligned} l_1^2+l_2^2+l_3^2+l_4^2&=1 \\ l_1m_1+l_2m_2+l_3m_3+l_4m_4&=0 \end{aligned}\right\}$$

Again multiplying the expressions for x_0, y_0, z_0, u_0 by l_1, l_2, l_3, l_4, respectively, and adding, we get, by virtue of these relations,

$$x=l_1x_0+l_2y_0+l_3z_0+l_4u_0,$$

and there are similar expressions for y, z, u.

If we square and add the new set of equations, and make $x^2+y^2+z^2+u^2=1$ for all values of x_0, y_0, z_0, u_0, we find another set of relations among the coefficients, of the types

$$\begin{aligned} l_1^2+m_1^2+n_1^2+p_1^2&=1 \\ l_1l_2+m_1m_2+n_1n_2+p_1p_2&=0. \end{aligned}$$

We may include all these equations in a scheme similar to that used for orthogonal transformation in ordinary geometry

	x	y	z	u
x_0	l_1	m_1	n_1	p_1
y_0	l_2	m_2	n_2	p_2
z_0	l_3	m_3	n_3	p_3
u_0	l_4	m_4	n_4	p_4

In this scheme, the sum of the squares of any row or column of the determinant is unity; and the sums of the products of the corresponding terms in any two rows, or in any two columns, is zero.

The square of the determinant by means of these relations reduces to unity, so that

$$\Delta^2=1.$$

If the positive directions of the edges of the tetrahedron retain the same relative positions towards each other, so that the tetrahedron could be moved into its new position, we must take $\Delta=+1$. This may be easily verified for simple cases. This is the only case that concerns us in the motion of a rigid body.

Comparing the equations

$$\left.\begin{aligned} \Delta x_0&=x_0L_1+y_0L_2+z_0L_3+p_0L_4 \\ x&=l_1x_0+l_2y_0+l_3y_0+l_4p_0 \end{aligned}\right\}$$

where L_1, L_2, L_3, L_4 are the minors of l_1, l_2, l_3, l_4, we see that each constituent in the determinant Δ is equal to its minor.

25. Let the coordinates of the edges 23, 31, 12, 14, 24, 34 be $(a, b, c, f, g, h)_{1,2,3,4,5,6}$. From the transformation of x_0, y_0, z_0, u_0 it is easily seen that

$$(y_0z'_0-y_0'z_0)=(yz'-zy')(m_2n_3-m_3n_2)+\ldots \text{ to six terms,}$$

that is

$$a_0 = aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1.$$

This and similar equations give us the formulæ for transforming the coordinates of a line. The transformation is orthogonal, and proceeding as before, we may use another transformation scheme, viz.

	a	b	c	f	g	h
a_0	a_1	b_1	c_1	f_1	g_1	h_1
b_0	a_2	b_2	c_2	f_2	g_2	h_2
c_0	a_3	b_3	c_3	f_3	g_3	h_3
f_0	a_4	b_4	c_4	f_4	g_4	h_4
g_0	a_5	b_5	c_5	f_5	g_5	h_5
h_0	a_6	b_6	c_6	f_6	g_6	h_6

This determinant possesses properties similar to those proved for the other determinant; the sum of the squares of the terms in any row, or any column, is unity, and the sum of the products of corresponding terms in any two rows, or in any two columns, is zero.

26. From this scheme it follows that the coordinates of the edges of the fixed tetrahedron referred to the other will be $(a_1, a_2, a_3, a_4, a_5, a_6)$, &c. There are other relations between the constituents of the determinant, due to the fact that opposite edges are conjugate polars with reference to the absolute quadric. Thus

$$\left.\begin{matrix} a_1 = f_4 \\ f_1 = a_4 \end{matrix}\right\} \quad \left.\begin{matrix} b_1 = g_4 \\ g_1 = b_4 \end{matrix}\right\} \quad \left.\begin{matrix} c_1 = h_4 \\ h_1 = c_4 \end{matrix}\right\}, \text{ \&c.}$$

If we square the determinant we get

$$D^2 = 1.$$

As before we take $D = 1$; in fact, D is the determinant formed out of the second minors of Δ. Whence

$$D = \Delta^3.*$$

Hence if we choose $\Delta = 1$ we must have $D = 1$ also.

Each constituent of the determinant D is equal to its minor. Also each minor of D is equal to its complementary minor.

27. In any displacement of a rigid body there are always two lines which remain in the same position after displacement.

* Cf. SCOTT's "Determinants," chap. v., § 9.

For let a, b, c, f, g, h be the coordinates of a line which remains unchanged. If such a line exist, we must have

$$\begin{array}{llllll}
0=a(a_1-1)+bb_1 & +cc_1 & +ff_1 & +gg_1 & +hh_1 \\
0=aa_2 & +b(b_2-1)+cc_2 & +ff_2 & +gg_2 & +hh_2 \\
0=aa_3 & +bb_3 & +c(c_3-1)+ff_3 & +gg_3 & +hh_3 \\
0=aa_4 & +bb_4 & +cc_4 & +f(f_4-1)+gg_4 & +hh_4 \\
0=aa_5 & +bb_5 & +cc_5 & +ff_5 & +g(g_5-1)+hh_5 \\
0=aa_6 & +bb_6 & +cc_6 & +ff_6 & +gg_6 & +h(h_6-1).
\end{array}$$

These equations are not independent; they are equivalent to four independent equations only.

We proceed to eliminate g and h between the equations 2, 3, and 4. The determinants thus introduced can be simplified by virtue of the relations between $(l, m, n, p)_{1,2,3,4}$. Thus

$$\begin{aligned}
(g_2h_3) &= (m_3p_1-m_1p_3)(n_1p_2-n_2p_1)-(m_1p_2-m_2p_1)(n_3p_1-n_1p_3) \\
&= p_1\{m_3(n_1p_2-n_2p_1)+m_2(n_3p_1-n_1p_3)+m_1(n_2p_3-n_3p_1)\} \\
&= -p_1l_4
\end{aligned}$$

Similarly

$$(g_3h_4)=p_1l_3 \qquad (g_4h_2)=-p_1l_2.$$

Again

$$\begin{aligned}
& a_4(g_2h_3)+a_2(g_3h_4)+a_3(g_4h_2) \\
&= -p_1\{l_4(m_1n_4-m_4n_1)+l_3(m_1n_3-m_3n_1)+l_2(m_1n_2-m_2n_1)\} \\
&= \quad p_1\{m_1.l_1n_1-n_1.l_1m_1\}=0.
\end{aligned}$$

Similarly

$$\begin{aligned}
b_4(g_2h_3) \quad +(b_2-1)(g_3h_4)+b_3(g_4h_2) &= -p_1(n_1+l_3) \\
c_4(g_2h_3) \quad +c_2(g_3h_4) \quad +(c_3-1)(g_4h_2) &= \quad p_1(l_2+m_1) \\
(f_4-1)(g_2h_3)+f_2(g_3h_4) \quad +f_3(g_4h_2) &= \quad p_1(l_4+p_1).
\end{aligned}$$

Hence (finally)

$$-p_1b(n_1+l_3)+p_1c(l_2+m_1)+p_1f(l_4+p_1)=0.$$

By a similar process we arrive at the following equations

$$\left.\begin{array}{l}
\quad . \qquad +b(n_1+l_3)-c(l_2+m_1)-f(l_4+p_1)=0 \\
-a(m_3+n_2) \quad . \qquad +c(l_2+m_1)-g(m_4+p_2)=0 \\
\;\; a(m_3+n_2)-b(n_1+l_3) \quad . \qquad -h(n_4+p_3)=0 \\
\;\; f(l_4+p_1)+g(m_4+p_2)+h(n_4+p_3) \quad . \qquad =0
\end{array}\right\}$$

Again, since $a_4=f_1$, $b_4=g_1$, $c_4=h_1$, &c., the last three of the original equations are the same as the first three, with f, g, h written respectively for a, b, c. Hence these three give rise to the group of equations

$$\begin{array}{llll} \cdot & +g(n_1+l_3) & -h(l_2+m_1) & -a(l_4+p_1)=0 \\ -f(m_3+n_2) & \cdot & +h(l_2+m_1) & -b(m_4+p_2)=0 \\ f(m_3+n_2) & -g(n_1+l_3) & \cdot & -c(n_4+p_3)=0 \\ a(l_4+p_1) & +b(m_4+p_2) & +c(n_4+p_3) & \cdot \quad =0 \end{array}$$

It is not difficult to show that the first three equations of the first group, with the last of this group, contain all the rest.

Besides these we have the equation

$$af+bg+ch=0.$$

If we substitute for f, g, h, their values in terms of a, b, c, it becomes

$$(l_4+p_1)bc\{(n_4+p_3)(l_2+m_1)-(m_4+p_2)(n_1+l_3)\}+\text{two similar terms}=0.$$

This with the equation

$$a(l_4+p_1)+b(m_4+p_2)+c(n_4+p_3)=0$$

will give us a quadratic in the ratios of $a:b$.

Hence there are two lines which remain fixed during any displacement. Since the equations to find f, g, h are exactly the same as those to find a, b, c it follows that one of the lines will be the polar of the other.

The above work implies that the determinant

$$\begin{vmatrix} a_1-1, & b_1, & c_1, & \dots \\ a_2, & b_2-1, & c_2, & \dots \\ a_3, & b_3, & c_3-1, & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

vanishes. I have verified by actual calculation and reduction in terms of $(l, m, n, p)_{1,2,3,4}$ that the determinant and all its first minors vanish; but the work is too long to be reproduced here.

Expressing the fact that two lines remain fixed during any displacement in kinematical language we learn that any displacement whatever of a rigid body may be produced by two rotations, about two lines which are conjugate to each other with reference to the absolute. Instead of rotations we might have said translations; or, expressing one of the rotations as a translation along the polar line of its axis, we learn that any displacement of a rigid body may be effected by a rotation about a line and a simultaneous translation along it. Instead of the line we might equally well have used its polar.

28. Suppose the position of any rigid body determined by the coordinates of the angular points of a quadrantal tetrahedron fixed in the body. Then, for a displacement of the body, we have the equations

$$\left.\begin{aligned}\dot{x}&=\dot{l}_1x_0+\dot{l}_2y_0+\dot{l}_3z_0+\dot{l}_4u_0\\ \dot{y}&=\dot{m}_1x_0+\dot{m}_2y_0+\dot{m}_3z_0+\dot{m}_4u_0\\ \dot{z}&=\dot{n}_1x_0+\dot{n}_2y_0+\dot{n}_3z_0+\dot{n}_4u_0\\ \dot{u}&=\dot{p}_1x_0+\dot{p}_2y_0+\dot{p}_3z_0+\dot{p}_4u_0\end{aligned}\right\}$$

The sixteen quantities $\dot{l}_1$, $\dot{l}_2$, &c., have relations among them, and it is possible to express them all in terms of six properly chosen variables. These new variables ω_1, ω_2, ω_3, ω_4, ω_5, ω_6 are thus defined :—

$$\begin{aligned}\omega_1&=l_1\dot{p}_1+l_2\dot{p}_2+l_3\dot{p}_3+l_4\dot{p}_4\\ \omega_2&=m_1\dot{p}_1+m_2\dot{p}_2+m_3\dot{p}_3+m_4\dot{p}_4\\ \omega_3&=n_1\dot{p}_1+n_2\dot{p}_2+n_3\dot{p}_3+n_4\dot{p}_4\\ \omega_4&=m_1\dot{n}_1+m_2\dot{n}_2+m_3\dot{n}_3+m_4\dot{n}_4\\ \omega_5&=n_1\dot{l}_1+n_2\dot{l}_2+n_3\dot{l}_3+n_4\dot{l}_4\\ \omega_6&=l_1\dot{m}_1+l_2\dot{m}_2+l_3\dot{m}_3+l_4\dot{m}_4\end{aligned}$$

Since

$$l_1p_1+l_2p_2+l_3p_3+l_4p_4=0 \text{ \&c.}$$

we immediately deduce six other equations by differentiation. These are

$$\left.\begin{aligned}-\omega_1&=\dot{l}_1p_1+\dot{l}_2p_2+\dot{l}_3p_3+\dot{l}_4p_4\\ -\omega_2&=\dot{m}_1p_1+\dot{m}_2p_2+\dot{m}_3p_3+\dot{m}_4p_4\\ -\omega_3&=\dot{n}_1p_1+\dot{n}_2p_2+\dot{n}_3p_3+\dot{n}_4p_4\\ -\omega_4&=\dot{m}_1n_1+\dot{m}_2n_2+\dot{m}_3m_3+\dot{m}_4m_4\\ -\omega_5&=\dot{n}_1l_1+\dot{n}_2l_2+\dot{n}_3l_3+\dot{n}_4l_4\\ -\omega_6&=\dot{l}_1m_1+\dot{l}_2m_2+\dot{l}_3m_3+\dot{l}_4m_4\end{aligned}\right\}$$

And besides these, we have the four equations

$$\begin{aligned}0&=l_1\dot{l}_1+l_2\dot{l}_2+l_3\dot{l}_3+l_4\dot{l}_4\\ 0&=m_1\dot{m}_1+m_2\dot{m}_2+m_3\dot{m}_3+m_4\dot{m}_4\\ 0&=n_1\dot{n}_1+n_2\dot{n}_2+n_3\dot{n}_3+n_4\dot{n}_4\\ 0&=p_1\dot{p}_1+p_2\dot{p}_2+p_3\dot{p}_3+p_4\dot{p}_4.\end{aligned}$$

From these it follows at once that

$$-\omega_1 . p_1 - \omega_6 m_1 + \omega_5 n_1 + 0 . l_1 = \dot{l}_1$$

In this way we arrive at the following equations :—

$$\left.\begin{array}{l} \dot{l}_1 = \quad . \quad -\omega_6 m_1 + \omega_5 n_1 - \omega_1 p_1 \\ \dot{m}_1 = \quad \omega_6 l_1 \quad . \quad -\omega_4 n_1 - \omega_2 p_1 \\ \dot{n}_1 = -\omega_5 l_1 + \omega_4 m_1 \quad . \quad -\omega_3 p_1 \\ \dot{p}_1 = \quad \omega_1 l_1 + \omega_2 m_1 + \omega_3 p_1 \quad . \end{array}\right\}$$

and similar equations hold for the other suffixes.

29. Substitute these values in the equations for $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{u}$, and make the variable tetrahedron instantaneously coincide with the fixed tetrahedron; then

$$\left.\begin{array}{l} \dot{x} = \quad . \quad -\omega_6 y + \omega_5 z - \omega_1 u \\ \dot{y} = \quad \omega_6 x \quad . \quad -\omega_5 z - \omega_2 u \\ \dot{z} = -\omega_5 x + \omega_4 y \quad . \quad -\omega_3 u \\ \dot{u} = \quad \omega_1 x + \omega_2 y + \omega_3 z \quad . \end{array}\right\}$$

To interpret the ω's suppose all except one (say ω_1) to vanish. Then $\dot{y}$, $\dot{z}$ vanish; therefore the distances of the moving point from the angular points 2 and 3 are constant. Hence the displacement is a rotation about the axis 23. If we put $u=1$, we get $\dot{x}=-\omega_1$. Hence ω_1 is the angular velocity about the axis 23. This angular velocity is from the angular point 1, towards the angular point 4.

Assuming the principle of superposition of small motions, we may say that the equations give us the rates of change of the variables x, y, z, u, due to a motion, which is the resultant of angular velocities ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, about the six edges of the fixed tetrahedron, 23, 31, 12, 14, 24, 34 respectively.

30. In exactly the same way we may express the variations of l, m, n, p, in terms of six other variables, θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, defined as follows :—

$$\left.\begin{array}{l} \theta_1 = \dot{l}_1 l_4 + \dot{m}_1 m_4 + \dot{n}_1 n_4 + \dot{p}_1 p_4 \\ \theta_2 = \dot{l}_2 l_4 + \dot{m}_2 m_4 + \dot{n}_2 n_4 + \dot{p}_2 p_4 \\ \theta_3 = \dot{l}_3 l_4 + \dot{m}_3 m_4 + \dot{n}_3 n_4 + \dot{p}_3 p_4 \\ \theta_4 = \dot{l}_2 l_3 + \dot{m}_2 m_3 + \dot{n}_2 n_3 + \dot{p}_2 p_3 \\ \theta_5 = \dot{l}_3 l_1 + \dot{m}_3 m_1 + \dot{n}_3 n_1 + \dot{p}_3 p_1 \\ \theta_6 = \dot{l}_1 l_2 + \dot{m}_1 m_2 + \dot{n}_1 n_2 + \dot{p}_1 p_2 \end{array}\right\}$$

These equations give us

$$\left.\begin{aligned}\dot{l}_1 &= \quad . \quad +\theta_6 l_2-\theta_5 l_3+\theta_1 l_4\\ \dot{l}_2 &= -\theta_6 l_1 \quad . \quad +\theta_4 l_3+\theta_2 l_4\\ \dot{l}_3 &= \theta_5 l_1-\theta_4 l_2 \quad . \quad +\theta_3 l_4\\ \dot{l}_4 &= -\theta_1 l_1-\theta_2 l_2-\theta_3 l_3 \quad .\end{aligned}\right\}$$

and similar equations in m, n, and p.

It will be seen that $-\theta_1$, $-\theta_2$, $-\theta_3$, $-\theta_4$, $-\theta_5$, $-\theta_6$, are formed from the coordinates (l_1, l_2, l_3, l_4), &c., in exactly the same way as ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, were formed from (l, m, n, p), &c. Hence, if we regard the variable tetrahedron as fixed, the other tetrahedron will have angular velocities $-\theta_1$, $-\theta_2$, $-\theta_3$, $-\theta_4$, $-\theta_5$, $-\theta_6$ relative to it. Hence, reversing the motion, we see that θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 are the angular velocities of the moving tetrahedron about fixed axes instantaneously coinciding with its own.

31. By definition,

$$\theta_1=\dot{l}_1 l_4+\dot{m}_1 m_4+\dot{n}_1 n_4+\dot{p}_1 p_4.$$

Substitute the values of $\dot{l}_1$, $\dot{m}_1$, $\dot{n}_1$, $\dot{p}_1$ in terms of the ω's; then

$$\begin{aligned}\theta_1 &= \omega_1(l_1p_4-l_4p_1)+\omega_2(m_1p_4-m_4p_1)+\omega_3(n_1p_4-n_4p_1)\\ &\qquad +\omega_4(m_1n_4-m_4n_1)+\omega_5(n_1l_4-n_4l_1)+\omega_6(l_1m_4-l_4m_1)\\ &= \omega_1 f_4+\omega_2 g_4+\omega_3 h_4+\omega_4 a_4+\omega_5 b_4+\omega_6 c_4\end{aligned}$$

i.e.,

$$\theta_1=\omega_1 a_1+\omega_2 b_1+\omega_3 c_1+\omega_4 f_1+\omega_5 g_1+\omega_6 h_1$$

Similarly for the other θ's. Hence the θ's can be expressed in terms of the ω's by the same transformation scheme that was used for expressing a_0, b_0, c_0, f_0, g_0, h_0, in terms of a, b, c, f, g, h.

32. This result gives us the law by which angular velocities and translations are resolved. For suppose all the θ's zero except θ_1, then

$$\left.\begin{aligned}\omega_1&=a_1\theta_1\\ \omega_2&=b_1\theta_1\\ \omega_3&=c_1\theta_1\end{aligned}\right\} \qquad \left.\begin{aligned}\omega_4&=f_1\theta_1\\ \omega_5&=g_1\theta_1\\ \omega_6&=h_1\theta_1\end{aligned}\right\}$$

In other words, any angular velocity θ, about a fixed line (a, b, c, f, g, h) is equivalent to component angular velocities $a\theta$, $b\theta$, $c\theta$, $f\theta$, $g\theta$, $h\theta$ about the six edges of the fundamental tetrahedron respectively.

A translation along a line may be resolved into translations along the six edges according to the same law.

Hence, the component angular velocity about a line $(a_1, b_1, c_1, f_1, g_1, h_1)$ of an angular velocity θ about (a, b, c, f, g, h) is

$$\theta(aa_1+bb_1+cc_1+ff_1+gg_1+hh_1)$$

If the lines meet, this law is similar to the parallelogrammic law in ordinary space. To find the component angular velocity about any line, we multiply the given angular velocity by the cosine of the inclination of the line to the axis of rotation.

Also the component translational velocity along the line $(a_1, b_1, c_1, f_1, g_1, h_1)$ due to the same angular velocity θ, is

$$\theta(af_1+bg_1+ch_1+fa_1+gb_1+hc_1)$$

Hence the expression $af_1+bg_1+ch_1+fa_1+gb_1+hc_1$ denotes the velocity along one line due to a unit angular velocity about the other. It is sometimes called the moment of the two lines.

33. We shall next find the rates of change of the coordinates of a line, due to the angular velocities ω_1, ω_2, ω_3, ω_4, ω_5, ω_6.

Take two points (x, y, z, u), (x', y', z', u'), distant a right angle from each other. We already know expressions for $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{u}$ in terms of the ω's. Now

$$\begin{aligned}\dot{a}&=\frac{d}{dt}(yz'-y'z)\\&=\dot{y}z'-\dot{z}y'+y\dot{z}'-z\dot{y}'\\&=(x\omega_6-z\omega_4-u\omega_2)z'-(-x\omega_5+y\omega_4-u\omega_3)y'\\&\quad+(-x'\omega_5+y'\omega_4-u'\omega_3)y-(x'\omega_6-z'\omega_4-u'\omega_2)z.\\&=\omega_6(xz'-x'z)-\omega_2(uz'-u'z)+\omega_5(xy'+x'y)-\omega_3(yu'-y'u)\end{aligned}$$

and therefore

$$\dot{a}=-\omega_6 b+\omega_5 c-\omega_3 g+\omega_2 h$$

Hence finally

$$\left.\begin{array}{lcccccc}\dot{a}= & . & -\omega_6 b & +\omega_5 c & . & -\omega_3 g & +\omega_2 h\\ \dot{b}= & \omega_6 a & . & -\omega_4 c & +\omega_3 f & . & -\omega_1 h\\ \dot{c}= & -\omega_5 a & +\omega_4 b & . & -\omega_2 f & +\omega_1 g & .\\ \dot{f}= & . & -\omega_3 b & +\omega_2 c & . & -\omega_6 g & +\omega_5 h\\ \dot{g}= & \omega_3 a & . & -\omega_1 c & +\omega_6 f & . & +\omega_4 h\\ \dot{h}= & -\omega_2 a & +\omega_1 b & . & -\omega_5 f & +\omega_4 g & .\end{array}\right\}$$

Since $-\theta_1$, $-\theta_2$, &c., correspond to ω_1, ω_2, &c., if we apply these formulæ to the edges of the fixed tetrahedron we find

$$
\begin{array}{llllll}
\dot{a}_1= & \cdot\ +\theta_6a_2-\theta_5a_3 & \cdot\ +\theta_3a_5-\theta_2a_6 \\
\dot{a}_2=-\theta_6a_1 & \cdot\ +\theta_4a_3-\theta_3a_4 & \cdot\ +\theta_1a_6 \\
\dot{a}_3=\ \theta_5a_1-\theta_4a_2 & \cdot\ +\theta_2a_4-\theta_1a_5 & \cdot \\
\dot{a}_4= & \cdot\ +\theta_3a_2-\theta_2a_3 & \cdot\ +\theta_6a_5-\theta_5a_6 \\
\dot{a}_5=-\theta_3a_1 & \cdot\ +\theta_1a_3-\theta_6a_4 & \cdot\ +\theta_4a_6 \\
\dot{a}_6=\ \theta_2a_1-\theta_1a_2 & \cdot\ +\theta_5a_4-\theta_4a_5 & \cdot
\end{array}
$$

34. Let $q_1, q_2, q_3, q_4, q_5, q_6$, be any quantities which obey the same law of resolution as has been proved for angular velocities; we can now find the rates of change of these quantities, referred to axes moving with the body. Let $Q_1, Q_2, \ldots Q_6$, be the rates of change of these quantities. Now $a_1q_1+a_2q_2+a_3q_3+a_4q_4+a_5q_5+a_6q_6$ is the component of the q's in a fixed direction. Hence we must have

$$\frac{d}{dt}(a_1q_1+a_2q_2+\ldots+a_6q_6)=a_1Q_1+a_2Q_2+\ldots+a_6Q_6$$

Differentiate out the left side, and substitute for $\dot{a}_1, \dot{a}_2, \dot{a}_3, \ldots \dot{a}_6$, in terms of the angular velocities of the moving axes about fixed axes instantaneously coinciding with them, *i.e.*, in terms of the θ's. Then

$$
\begin{aligned}
&a_1Q_1+a_2Q_2+\ldots+a_6Q_6\\
&=a_1(\dot{q}_1-\theta_6q_2+\theta_5q_3-\theta_3q_5+\theta_2q_6)\\
&+\text{ similar terms in } a_2, a_3 \ldots a_6,
\end{aligned}
$$

and there are similar equations with $b, c, \ldots$ substituted for a. If we multiply these equations by $a_1, b_1, c_1, \ldots$, and add, we find an expression for Q_1. In this way we arrive at the equations

$$
\begin{array}{llll}
Q_1=\dot{q}_1 & \cdot\ -\theta_6q_2+\theta_5q_3 & \cdot\ -\theta_3q_5+\theta_2q_6 \\
Q_2=\dot{q}_2+\theta_6q_1 & \cdot\ -\theta_4q_3+\theta_3q_4 & \cdot\ -\theta_1q_6 \\
Q_3=\dot{q}_3-\theta_5q_1+\theta_4q_2 & \cdot\ -\theta_2q_4+\theta_1q_5 & \cdot \\
Q_4=\dot{q}_4 & \cdot\ -\theta_3q_2+\theta_2q_3 & \cdot\ -\theta_6q_5+\theta_5q_6 \\
Q_5=\dot{q}_5+\theta_3q_1 & \cdot\ -\theta_1q_3+\theta_6q_4 & \cdot\ -\theta_4q_6 \\
Q_6=\dot{q}_6-\theta_2q_1+\theta_1q_2 & \cdot\ -\theta_5q_4+\theta_4q_5 & \cdot
\end{array}
$$

Theory of screws.

35. It has been shown that any state of motion of a rigid body may be reduced to a translation along a line and a simultaneous rotation about it. Such a motion is

called a twist about a certain screw whose axis is the given line. The same motion may be effected by a twist on a screw whose axis is the polar line. Let the angular velocities about the edges of the fixed quadrantal tetrahedron due to any such twist be Ωa, Ωb, Ωc, Ωf, Ωg, Ωh, respectively, where

$$a^2+b^2+c^2+f^2+g^2+h^2=1.$$

Then a, b, c, f, g, h, may be defined to be the coordinates of the screw, and Ω the magnitude of the twist about it. The quantity $2\{af+bg+ch\}$ may be called the parameter of the screw, and will be denoted by k. If $k=0$, the coordinates are the coordinates of a line, and the motion degenerates into a rotation about the line represented by the coordinates.

The locus of lines which have no lengthwise velocity due to a twist about the screw, is the linear complex

$$\mathrm{a}f+\mathrm{b}g+\mathrm{c}h+\mathrm{f}a+\mathrm{g}b+\mathrm{h}c=0.$$

This we shall call (after LINDEMANN) the Rotation Complex. If Ωa, Ωb . . . had been translations instead of rotations, the same property would have been enjoyed by the polar complex

$$\mathrm{a}a+\mathrm{b}b+\mathrm{c}c+\mathrm{f}f+\mathrm{g}g+\mathrm{h}h=0.$$

This is called the Translation Complex. Any property of one complex has a corresponding property of the other.

The trajectory of every point on a line of the rotation complex is perpendicular to the line. Hence the trajectory of any point whatever is normal to the plane which corresponds to the point in the complex.

If we refer back to the expressions for $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{f}$, $\dot{g}$, $\dot{h}$, we see that

and also
$$\left.\begin{aligned}\mathrm{a}\dot{f}+\mathrm{b}\dot{g}+\mathrm{c}\dot{h}+\mathrm{f}\dot{a}+\mathrm{g}\dot{b}+\mathrm{h}\dot{c}=0\\ \mathrm{a}\dot{a}+\mathrm{b}\dot{b}+\mathrm{c}\dot{c}+\mathrm{f}\dot{f}+\mathrm{g}\dot{g}+\mathrm{h}\dot{h}=0\end{aligned}\right\}$$

Hence, if any line belongs to either complex, it will belong to it after receiving a small twist about the corresponding screw. Hence, by superimposing such small twists, it follows that both complexes are transformed into themselves by a twist about the corresponding screw.

36. The screw may be replaced by two rotations, or two translations, in an infinite number of ways; any line may be taken as one axis of rotation or translation, the other axis being the conjugate polar of the first with respect to the Rotation Complex.

For let the twist about the given screw be equivalent to rotations λ, μ, about axes a_1, b_1, c_1, f_1, g_1, h_1, and a_2, b_2, c_2, f_2, g_2, h_2, respectively.

Then

$$\left.\begin{aligned}\Omega \mathrm{a}&=\lambda a_1+\mu a_2\\ \Omega \mathrm{b}&=\lambda b_1+\mu b_2\\ \Omega \mathrm{c}&=\lambda c_1+\mu c_2\end{aligned}\right\}\qquad \left.\begin{aligned}\Omega \mathrm{f}&=\lambda f_1+\mu f_2\\ \Omega \mathrm{g}&=\lambda g_1+\mu g_2\\ \Omega \mathrm{b}&=\lambda h_1+\mu h_2\end{aligned}\right\}$$

Squaring and adding we get

$$\Omega^2=\lambda^2+\mu^2+2\lambda\mu\cos\theta\cos\delta;$$

also

$$\Omega^2 k=\lambda\mu\sin\theta\sin\delta$$

where δ is the length of a common perpendicular to the two lines and θ the angle between them. Further, from the form of the above equations, it follows that if any line of the rotation complex meet one axis of rotation, it will meet the other. Hence the two axes of rotation are conjugate polars with respect to the rotation complex.

The axes of the screw are obtained by making the second line the polar of the first. This gives

$$\left.\begin{aligned}\Omega \mathrm{a}&=\lambda a_1+\mu f_1\\ \Omega \mathrm{f}&=\lambda f_1+\mu a_1\end{aligned}\right\}\ \&c.$$

and therefore

$$\left.\begin{aligned}\lambda^2+\mu^2&=\Omega^2\\ \lambda\mu&=\Omega^2 k\end{aligned}\right\}$$

The axes of the screw are conjugate polars with respect to both complexes; *i.e.*, they are the directrices of the congruance formed by lines common to the two complexes.

37. We shall now find the surface corresponding to the Cylindroid.

Let one twist be defined by the components $\Omega \mathrm{a}$, $\Omega \mathrm{f}$, the others being zero; and a second twist by the components $\Omega' \mathrm{b}'$, $\Omega' \mathrm{g}'$, the rest being zero. Let the resultant motion be a rotation λ about a line a, b, c, f, g, h, and a translation μ along it; we have to find the surface generated by this line. As before

$$\left.\begin{aligned}\Omega \mathrm{a}&=\lambda a+\mu f\\ \Omega \mathrm{f}&=\lambda f+\mu a\end{aligned}\right\}\qquad \left.\begin{aligned}\Omega' \mathrm{b}'&=\lambda b+\mu g\\ \Omega' \mathrm{g}'&=\lambda g+\mu b\end{aligned}\right\}$$

and

$$c=0,\qquad h=0.$$

Hence the new axis always meets the lines (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1), *i.e.*, the new axis always meets the common perpendiculars to the axes of the given screws.

From these equations we easily deduce

$$\lambda(af\ -fa\ )=\mu(aa\ -ff\ )$$
$$\lambda(bg'-gb')=\mu(bb'-gg')$$

Hence eliminating $\lambda : \mu$

$$(b'f-ag')(ab-fg)=(ab'-fg')(bf-ag).$$

This equation must be expressed in point coordinates. Since $c=0$ and $h=0$,

$$\frac{x'}{x}=\frac{y'}{y}=m \text{ say},$$

and

$$\frac{z'}{z}=\frac{u'}{u}=n \text{ say}.$$

Substituting for x', y', z', u' in the expressions for a, b, f, g, the factor $(m-n)^2$ divides out and the equation to the Cylindroid becomes finally

$$(b'f-ag')xy(z^2+u^2)=(ab'-fg')zu(x^2+y^2).$$

This is a surface of the fourth order having the common perpendiculars of the axes of the two given screws, for nodal lines.

38. To find the conditions that any number of twists about given screws may produce rest.

The method here employed is the same as that given by SPOTTISWOODE, in the 'Comptes Rendus,' t. lxvi.

Let the coordinates of the screws be $a_0, b_0 \ldots a_1, b_1 \ldots$, there being n screws; and let the magnitudes of the twists on them be $\Omega_0, \Omega_1 \ldots \Omega_{n-1}$. Then the conditions that they will neutralise each other will be

$$\Sigma(\Omega a)=0,\ \Sigma(\Omega b)=0\ .\ .\ .\ \Sigma(\Omega h)=0.$$

The expression

$$a_0f_1+b_0g_1+c_0h_1+f_0a_1+g_0b_1+h_0c_1$$

is called the simultaneous invariant of the two screws 0 and 1, and it will be denoted by (01). Similar expressions will apply to the other screws. The quantity (00) will be the parameter of the screw (0).

By means of the equations of condition we get

$$\Sigma(\Omega a)\Sigma(\Omega f)+\Sigma(\Omega b)\Sigma(\Omega g)+\Sigma(\Omega c)\Sigma(\Omega h)=0.$$

Multiply these out, then

$$\Sigma\Sigma\Omega_i\Omega_j(i,j)+\tfrac{1}{2}\Sigma\Omega_i^2(i,i)=0.$$

Here Σ implies summation from 0 to $\overline{n-1}$ inclusive.

Now let Σ' imply summation from 1 to $\overline{n-1}$ inclusive. Then writing the equations of condition in the form

$$\Sigma'(\Omega a)=-\Omega_0 a_0, \text{ \&c.},$$

we can show, as before, that

$$\Sigma'(\Omega a)\Sigma'(\Omega f)+\ldots+\ldots=\tfrac{1}{2}\Omega_0^2(00),$$

i.e.,

$$\Sigma'\Sigma'\Omega_i\Omega_j(i,j)+\tfrac{1}{2}\Sigma'\Omega_i^2(i,i)=\tfrac{1}{2}\Omega_0^2(00).$$

Subtract this from the former equation, then

$$\Sigma'\Omega_1\Omega_0(0,1)+\Omega_0^2(0,0)=0.$$

In this way we deduce the following equations

$$\begin{array}{l}\Omega_0(0,0)+\Omega_1(0,1)+\Omega_2(0,2)+\ldots=0\\ \Omega_0(1,0)+\Omega_1(1,1)+\Omega_2(1,2)+\ldots=0\\ \Omega_0(2,0)+\Omega_1(2,1)+\Omega_2(2,2)+\ldots=0\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}$$

The condition that these should be simultaneously satisfied is that the determinant should vanish; that is,

$$\begin{vmatrix}(0,0) & (0,1) & (0,2) & \ldots\\ (1,0) & (1,1) & (1,2) & \ldots\\ (2,0) & (2,1) & (2,2) & \ldots\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{vmatrix}=0.$$

39. Since $(0,1)=(1,0)$, &c., this determinant is symmetrical. Let $[0,0]$, $[0,1]$, ... denote the coefficients of $(0,0)$, $(0,1)$, ... in the expansion of the determinant. Then $[0,1]=[1,0]$.

Then we can solve any $(n-1)$ of the equations to find the ratios $\Omega_0 : \Omega_1 : \Omega_2 \ldots$ Suppressing the first equation we get

$$\Omega_0 : [0,0]=(-1)^{n-1}\Omega_1 : [0,1]$$

Similarly, by suppressing the second equation, we find that

$$\Omega_0 : [1,0]=(-1)^{n-1}\Omega_1 : [1,1]$$

and so on. Hence finally

$$\Omega_0^2 : \Omega_1^2 : \Omega_2^2 \ldots=[0,0] : [1,1] : [2,2] \ldots$$

The determinant equated to zero gives the relation between the coordinates of the screws, and these final equations determine the ratios of the twists on them in order that they may produce rest.

In the case of simple rotations or translations the coordinates become the coordinates of lines, and therefore,

$$(0,0)=0 \qquad (1,1)=0 \ldots$$

(0,1) will be the moment of the lines of action of the translations or rotations, and the conditions that the system may produce rest are of the same form as before.

Kinetics.

40. The definitions of acceleration, momentum, and kinetic energy of a particle are taken to be exactly the same as in ordinary space, and do not need further comment.

We shall next consider the measure of force. To determine a force completely we require a line of action and a measure of its magnitude. We assume that a force may be applied at any point of its line of action. As in ordinary space we shall take NEWTON'S second law of motion as the basis of the measurement of forces. This law states that "change of motion is proportional to the impressed force, and takes place in the direction in which the force acts." Hence, if a force act on a particle of known mass it will cause a change of momentum in the direction of its action, and the measure of the force will be proportional to the change of momentum per unit of time. By the proper choice of the unit of force we may deduce the equation

$$P=mf$$

which gives the dynamical measure of any force.

Since a linear acceleration f along any line is exactly the same thing as an angular acceleration f about the polar line, it follows that a force P along any line is the same thing as a couple P about the polar line.

41. NEWTON'S law further implies the physical independence of forces; *i.e.*, if a number of forces act on a body each produces the same effect as if it alone acted on the body. Hence if a number of forces act on any number of particles each force produces its own effect, and to calculate the resultant of the forces we must calculate the resultant change of momentum per unit time; in other words, the laws by which forces are combined are the same as those for velocities and accelerations. If, therefore, a force P act along a given line whose coordinates are a, b, c, f, g, h its effect is the same as if forces Pa, Pb, . . . acted along the edges of the tetrahedron of reference; and the component force along any other line a', b', c', f', g', h' is

$$P(aa'+bb'+cc'+ff'+gg'+hh')$$

and the couple about it is

$$P(af'+bg'+ch'+fa'+gb'+ch').$$

42. All the theorems about compounding linear and rotational velocities apply to systems of forces and couples. Thus any system of forces may be reduced to two, acting along lines which are conjugate to each other with respect to a certain linear complex. Any system of forces may be reduced to a wrench about a certain screw.

We can now give an interpretation to the simultaneous invariant of two screws. For suppose Ω is a twist about a screw a, b, c, . . . , and Q a wrench about another screw a′, b′, c′, . . . Then Ωa, Ωb, . . . are angular velocities about the six edges of the tetrahedron of reference, and Qa′, Qb′, . . . are couples about them. Hence if the body receive a small displacement about the first screw, while the wrench Q about the other screw is acting on the body, the rate of doing work will be

$$\Omega Q(af'+bg'+ch'+fa'+gb'+hc')$$

and therefore the simultaneous invariant of two screws is the rate of doing work, when the body has a unit twist about one screw, while a unit wrench on the other screw is acting on the body.

43. The condition of equilibrium of any number of wrenches on given screws is the same as the condition that a number of twists about the screws should neutralise each other. If the wrenches reduce to couples or forces we have only to make $(0,0)=0$, $(1,1)=0$, . . . The conditions, therefore, for the equilibrium of any number of forces P_0, P_1, . . . acting along given lines are, first, that the determinant

$$\begin{vmatrix} \cdot & (0,1) & (0,2) & \dots \\ (1,0) & \cdot & (1,2) & \dots \\ (2,0) & (2,1) & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

should vanish, and further, that the forces should be proportional to the square roots of the minors of the terms of the leading diagonal of this determinant.

44. Since $P=mf$, if we resolve the force P into the components $X_1, X_2, \dots X_6$ along the edges of the fixed quadrantal tetrahedron, and if $\dot{v}_1, \dot{v}_2, \dots \dot{v}_6$ be the components of f along these edges, we have

$$m\dot{v}_i=X_i \qquad (i=1,2,3,4,5,6).$$

Applying these equations to all the particles of the body, and using D'ALEMBERT'S principle, the equations of motion of the body become

$$\Sigma m\dot{v}_i=\Sigma X_i \qquad (i=1,2,3,4,5,6).$$

Multiply these equations by $v_1, v_2, \ldots v_6$ and add; then

$$\Sigma m(v_1\dot{v}_1+v_2\dot{v}_2+\ldots)=\Sigma(X_1v_1+\ldots),$$

and therefore integrating,

$$\tfrac{1}{2}\Sigma mv^2=\Sigma\int(X_1v_1+\ldots)dt,$$

which is the equation of conservation of energy.

45. We must now express the velocities of all the particles of the body in terms of the given angular velocities $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$.

Let a, b, c, f, g, h be the line through (x, y, z, u) along which this point begins to move. The line will therefore join the two points (x, y, z, u) and $(x+\dot{x}\delta t,\ y+\dot{y}\delta t,\ z+\dot{z}\delta t,\ u+\dot{u}\delta t)$. Hence

$$\begin{aligned} &a : b : c : f : g : h \\ &=y(z+\dot{z}\delta t)-z(y+\dot{y}\delta t) : \ldots \\ &=y\dot{z}-z\dot{y} : z\dot{x}-x\dot{z} : x\dot{y}-y\dot{x} : x\dot{u}-u\dot{x} : y\dot{u}-u\dot{y} : z\dot{u}-u\dot{z} \end{aligned}$$

But

$$y\dot{z}-z\dot{y}=zu\omega_2-yu\omega_3+(y^2+z^2)\omega_4-xy\omega_5-zx\omega_6$$

and we have similar expressions for the other terms. Hence

$$\begin{array}{llllll}
\mu f=(x^2+u^2)\omega_1 & +xy\omega_2 & +zx\omega_3 & \cdot & -zu\omega_5 & +yu\omega_6 \\
\mu g=xy\omega_1 & +(y^2+u^2)\omega_2 & +yz\omega_3 & +zu\omega_4 & \cdot & -xu\omega_6 \\
\mu h=zx\omega_1 & +yz\omega_2 & +(z^2+u^2)\omega_3 & -yu\omega_4 & +xu\omega_5 & \cdot \\
\mu a=\quad\cdot & +zu\omega_2 & -yu\omega_3 & +(y^2+z^2)\omega_4 & -xy\omega_5 & -zx\omega_6 \\
\mu b=-zu\omega_1 & \cdot & +xu\omega_3 & -xy\omega_4 & +(z^2+x^2)\omega_5 & -yz\omega_6 \\
\mu c=yu\omega_1 & -xu\omega_2 & \cdot & -zx\omega_4 & -yz\omega_5 & +(x^2+y^2)\omega_6
\end{array}$$

where μ is some common factor yet to be determined.

46. Square and add these expressions. The left side reduces to μ^2; on the right, the coefficients may be simplified by virtue of the relation, $x^2+y^2+z^2+u^2=1$. Thus the coefficient of $\omega_1{}^2$ is

$$\begin{aligned} &x^4+2x^2u^2+u^4+x^2y^2+z^2x^2+z^2u^2+y^2u^2 \\ &=(x^2+u^2)(x^2+y^2+z^2+u^2) \\ &=x^2+u^2. \end{aligned}$$

Again, the coefficent of $2\omega_1\omega_2$ is

$$xy(x^2+u^2)+xy(y^2+u^2)+xyz^2-xyu^2$$
$$=xy(x^2+y^2+z^2+u^2)$$
$$=xy.$$

All the other terms may be simplified in the same way, and therefore finally

$$\begin{aligned}\mu^2= \quad & \omega_1{}^2(x^2+u^2)+\omega_2{}^2(y^2+u^2)+\omega_3{}^2(z^2+u^2)\\ & +\omega_4{}^2(y^2+z^2)+\omega_5{}^2(z^2+x^2)+\omega_6{}^2(x^2+y^2)\\ & +2xy(\omega_2\omega_3-\omega_5\omega_6)+2zx(\omega_3\omega_1-\omega_6\omega_4)+2xy(\omega_1\omega_2-\omega_4\omega_5)\\ & +2xu(\omega_3\omega_5-\omega_2\omega_6)+2yu(\omega_1\omega_6-\omega_3\omega_4)+2zu(\omega_2\omega_4-\omega_1\omega_3)\end{aligned}$$

Next multiply the equations by $\omega_1, \omega_2, \ldots \omega_6$ and add; the same expression as before appears on the right, and therefore

$$\mu\{\omega_1 f+\omega_2 g+\omega_3 h+\omega_4 a+\omega_5 b+\omega_6 c\}=\mu^2,$$

that is

$$\mu=\omega_1 f+\omega_2 g+\omega_3 h+\omega_4 a+\omega_5 b+\omega_6 c.$$

Hence μ is the velocity of the point under consideration, and the above expressions for $\mu a, \mu b, \ldots$ are the velocities of the point (x, y, z, u) resolved along the six edges of the tetrahedron.

47. Multiplying them by m the mass of the particle, and taking the sum for all the particles of the body, we find expressions for the linear momenta of the body resolved along the six edges of the tetrahedron. For brevity, let $A=\Sigma m(x^2+u^2)$, &c., and let $P_1=\Sigma myz$, $P_2=\Sigma mzx$, &c. Then if $(q)_{1,2,3,4,5,6}$ denote the angular momenta about the edges 23, 31, 12, 14, 24, 34, or the linear momenta along the edges 14, 24, 34, 23, 31, 12 respectively,

$$\begin{aligned}q_1 &= A\omega_1+P_3\omega_2+P_2\omega_3 \quad . \quad -P_6\omega_5+P_5\omega_6\\ q_2 &= P_3\omega_1+B\omega_2+P_1\omega_3+P_6\omega_4 \quad . \quad -P_4\omega_6\\ q_3 &= P_2\omega_1+P_1\omega_2+C\omega_3-P_5\omega_4+P_4\omega_5 \quad .\\ q_4 &= \quad . \quad +P_6\omega_2-P_5\omega_3+F\omega_4-P_3\omega_5-P_2\omega_6\\ q_5 &= -P_6\omega_1 \quad . \quad +P_4\omega_3-P_3\omega_4+G\omega_5-P_1\omega_6\\ q_6 &= P_5\omega_1+P_4\omega_2 \quad . \quad -P_2\omega_4-P_1\omega_5+H\omega_6\end{aligned}$$

Also taking $\frac{1}{2}\Sigma m\mu^2$ we get the kinetic energy, viz.:

$$\begin{aligned}2T=&A\omega_1{}^2+B\omega_2{}^2+C\omega_3{}^2+F\omega_4{}^2+G\omega_5{}^2+H\omega_6{}^2\\ &+2P_1(\omega_2\omega_3-\omega_5\omega_6)+2P_2(\omega_3\omega_1-\omega_6\omega_4)+2P_3(\omega_1\omega_2-\omega_4\omega_5)\\ &+2P_4(\omega_3\omega_5-\omega_2\omega_6)+2P_5(\omega_1\omega_6-\omega_3\omega_4)+2P_6(\omega_2\omega_4-\omega_1\omega_3)\end{aligned}$$

48. The last article suggests the idea of moments of inertia. We may define the sum of the products of the masses of all the particles of a body into the squares of the sines of the perpendicular distances from a given plane, as the moment of inertia of the body with regard to the given plane. Similarly the product of inertia with regard to two planes is the sum of the masses multiplied by the product of the sines of the perpendicular distances from the planes. The moment of inertia about a line may be defined in a similar way. The moment of inertia of a body about any plane (l, m, n, p) is

$$\Sigma m(lx+my+nz+pu_2)^2$$

i.e.,

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2+2lm\Sigma mxy+\ldots$$

Thus the equation

$$l^2\Sigma mx^2+\ldots+2mn\Sigma yz+\ldots=0$$

is the tangential equation of a surface of the second order, which has the property that the moment of inertia about any tangent plane is zero. Professor CLIFFORD calls this the null surface of the body. The property just mentioned is independent of the system of coordinates chosen.

49. By an orthogonal transformation of such an equation of the second degree we can rid the equation of the products $m, n, \ldots$, and at the same time keep $l^2+m^2+n^2+p^2$ unchanged. The surface now becomes

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2=0.$$

The planes of the particular quadrantal tetrahedron chosen may be called the four principal planes of the body, and the edges the six principal axes of the body.

The envelope of planes which give a constant moment of inertia, MK^2, is given by the equation,

$$l^2\Sigma mx^2+m^2\Sigma my^2+n^2\Sigma mz^2+p^2\Sigma mu^2=MK^2(l^2+m^2+n^2+p^2).$$

This is a surface of the second order. Now

$$l^2+m^2+n^2+p^2=0$$

is the tangential equation to the Absolute quadric. Hence, this surface has the same common tangent planes with the Absolute as the null-surface has. In other words, this surface and the null-surface stand in the same relation to each other as confocal quadrics in ordinary geometry. Hence the planes which give a constant moment of inertia envelope a quadric confocal with the null-surface.

50. We next consider the moment of inertia about a line.

Let a, b, c, f, g, h, be the coordinates of any line, then it was shown in § 17, that if ϖ be the sine of the perpendicular distance from any point (x, y, z, u) to the line,

$$\begin{aligned}\varpi^2=&a^2(x^2+u^2)+b^2(y^2+u^2)+c^2(z^2+u^2)\\&+f^2(y^2+z^2)+g^2(z^2+x^2)+h^2(x^2+y^2)\\&+2yz(bc-gh)+2zx(ca-hf)+2xy(ab-fg)\\&+2xu(cg-bh)+2yu(ah-cf)+2zu(fb-ag).\end{aligned}$$

Making use of the previous notation the expression for the moment of inertia becomes

$$\begin{aligned}Mk^2=&Aa^2+Bb^2+Cc^2+Ff^2+Gg^2+Hh^2\\&+2P_1(bc-gh)+2P_2(ca-hf)+2P_3(ab-fg)\\&+2P_4(cg-bh)+2P_5(ah-cf)+2P_6(fb-ag).\end{aligned}$$

If we refer this to the principal tetrahedron of inertia, all the P's vanish, and it reduces to

$$Mk^2=Aa^2+Bb^2+Cc^2+Ff^2+Gg^2+Hh^2.$$

It must be noticed that the quantities A, B, C, F, G, H are not independent; for since

$$x^2+y^2+z^2+u^2=1,$$

we have

$$A+F=B+G=C+H=M,$$

where M is the mass of the body.

51. We are now in a position to write down the most general equations of motion of a rigid body, under the action of any forces, referred to a quadrantal tetrahedron moving in space, in any manner. For in § 34 we obtained formulæ for the rates of change of any quantities, obeying the usual law of resolution and composition, relatively to axes moving in any manner. Applying these formulæ we can obtain the rate of change of momentum relative to any moving axes. Equating the rate of change of momentum to the impressed forces, we get the equations of motion. Let $q_1, q_2, \ldots q_6$ denote the six components of angular momentum about the six edges of the tetrahedron, and let $Q_1, Q_2, \ldots Q_6$ denote the total impressed couples about these lines. The expressions for $q_1, q_2, \ldots q_6$, have been given in § 47. The equations of motion are, therefore,

$$\begin{aligned}&\dot{q}_1 \quad \cdot \quad +\theta_6q_2+\theta_5q_3 \quad \cdot \quad -\theta_3q_5+\theta_2q_6=Q_1\\&\dot{q}_2+\theta_6q_1 \quad \cdot \quad -\theta_4q_3+\theta_3q_4 \quad \cdot \quad -\theta_1q_6=Q_2\\&\dot{q}_3-\theta_5q_1+\theta_4q_2 \quad \cdot \quad -\theta_2q_4+\theta_1q_5 \quad \cdot \quad =Q_3\\&\dot{q}_4 \quad \cdot \quad -\theta_3q_2+\theta_2q_3 \quad \cdot \quad -\theta_6q_5+\theta_5q_6=Q_4\\&\dot{q}_5+\theta_1q_1 \quad \cdot \quad -\theta_1q_3+\theta_6q_4 \quad \cdot \quad -\theta_4q_6=Q_5\\&\dot{q}_6-\theta_2q_1+\theta_1q_2 \quad \cdot \quad -\theta_5q_4+\theta_4q_5 \quad \cdot \quad =Q_6\end{aligned}$$

We may write the values of the q's in the form

$$q_i=\left(\frac{d\mathrm{T}}{d\omega}\right)_i (i=1, 2, 3, 4, 5, 6.)$$

where T has the value given to it in § 47.

52. If we suppose the tetrahedron fixed in the body, we must put $\theta_1=\omega_1$, $\theta_2=\omega_2\ldots$; and if further, the tetrahedron be the principal tetrahedron of inertia, the equations will correspond to EULER's equations. They then become

$$\begin{aligned}
\mathrm{A}\dot{\omega}_1-(\mathrm{B}-\mathrm{H})\omega_2\omega_6-(\mathrm{G}-\mathrm{C})\omega_5\omega_3&=\mathrm{Q}_1\\
\mathrm{B}\dot{\omega}_2-(\mathrm{H}-\mathrm{A})\omega_6\omega_1-(\mathrm{C}-\mathrm{F})\omega_3\omega_4&=\mathrm{Q}_2\\
\mathrm{C}\dot{\omega}_3-(\mathrm{A}-\mathrm{G})\omega_1\omega_5-(\mathrm{F}-\mathrm{B})\omega_4\omega_2&=\mathrm{Q}_3\\
\mathrm{F}\dot{\omega}_4-(\mathrm{B}-\mathrm{C})\omega_2\omega_3-(\mathrm{G}-\mathrm{H})\omega_5\omega_6&=\mathrm{Q}_4\\
\mathrm{G}\dot{\omega}_5-(\mathrm{C}-\mathrm{A})\omega_3\omega_1-(\mathrm{H}-\mathrm{F})\omega_6\omega_4&=\mathrm{Q}_5\\
\mathrm{H}\dot{\omega}_6-(\mathrm{A}-\mathrm{B})\omega_1\omega_2-(\mathrm{F}-\mathrm{G})\omega_4\omega_5&=\mathrm{Q}_6.
\end{aligned}$$

53. If the body be moving under the action of no forces, there are two invariable complexes, viz.:

$$q_1f+q_2g+q_3h+q_4a+q_5b+q_6c=0$$

and its conjugate complex. For since the equations of motion express the fact that the q's do not change, these complexes will be fixed in space. The complex whose equation has just been given is the locus of lines along which there is no linear momentum. If no forces act, if we multiply the equations of motion respectively by $\omega_1, \omega_2, \ldots \omega_6$, and add, we get by integration

$$\mathrm{A}\omega_1^2+\mathrm{B}\omega_2^2+\mathrm{C}\omega_3^2+\mathrm{F}\omega_4^2+\mathrm{G}\omega_5^2+\mathrm{H}\omega_6^2=2\mathrm{T}$$

which is the equation of conservation of energy.

If again we multiply the equations of motion by $\mathrm{A}\omega_1$, $\mathrm{B}\omega_2$, . . . , and add, we arrive at another integral of the equations,

$$\mathrm{A}^2\omega_1^2+\mathrm{B}^2\omega_2^2+\mathrm{C}^2\omega_3^2+\mathrm{F}^2\omega_4^2+\mathrm{G}^2\omega_5^2+\mathrm{H}^2\omega_6^2=\mathrm{K}^2,$$

which expresses the fact that the sum of the squares of the q's is constant.

One more integral may easily be found. For let

$$\mathrm{A}=ma^2,\quad \mathrm{B}=mb^2,\ldots$$

then

$$f^2=1-a^2\quad g^2=1-b^2\quad h^2=1-c^2$$

and

$$\mathrm{G}-\mathrm{C}=\mathrm{M}(1-b^2-c^2)=\mathrm{H}-\mathrm{B}, \&c.$$

Hence the first three equations may be written in the forms

$$a^2\dot{\omega}_1+(1-b^2-c^2)(\omega_2\omega_6-\omega_5\omega_3)=0$$
$$b^2\dot{\omega}_2+(1-c^2-a^2)(\omega_3\omega_4-\omega_1\omega_6)=0$$
$$c^2\dot{\omega}_3+(1-a^2-b^2)(\omega_1\omega_5-\omega_2\omega_4)=0.$$

From these we immediately deduce the equation

$$\frac{a^2\omega_1^2}{1-b^2-c^2}+\frac{b^2\omega_2^2}{1-c^2-a^2}+\frac{c^2\omega_3^2}{1-a^2-b^2}=\text{a constant.}$$

Solution of the equations in terms of the double theta-functions.

54. The equations of motion of a solid body under the action of no forces can be solved in terms of the theta-functions of two variables, one of the variables being a linear function of the time. The solution, however, is not complete, owing to a deficiency in the number of arbitrary constants.

The notation employed will be that given by Mr. FORSYTH in his "Memoir on the Double Theta-functions," published in the Philosophical Transactions for 1882.

The definition of a double theta-function is expressed by the equation

$$\Phi\left\{\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix} x, y\right\}=\sum_{m=-\infty}^{m=\infty}\cdot\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{(2m+\mu)^2}{4}}q^{\frac{(2n+\nu)^2}{4}}r^{\frac{(2m+\mu)(2n+\nu)}{2}}v^{x(2m+\mu)}w^{y(2n+\nu)}$$

in which λ, μ, ρ, ν are given integers (afterwards taken to be each either zero or unity) and $\begin{pmatrix}\lambda, & \rho\\ \mu, & \nu\end{pmatrix}$ is called the characteristic; x, y are the variables; p, q, r, v, w are known constants, called parameters (in our case arbitrary constants); and the double summation extends to all positive and negative integral values (including zero) of m and n. There are sixteen different double theta-functions distinguished by suffixes 0, 1, 2, ... 15. These are written $\vartheta_0(x, y)$, $\vartheta_1(x, y)$, ..., or when the variables are easily understood, simply ϑ_0, ϑ_1, ...

The characteristics of the sixteen functions are given in the following table:—

$\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}0, & 0\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}$
ϑ_0	ϑ_2	ϑ_{10}	ϑ_8	ϑ_1	ϑ_3	ϑ_{11}	ϑ_9

$\begin{pmatrix}1, & 0\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 1, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 1, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 0, & 0\end{pmatrix}$	$\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}$	$\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix}$
ϑ_5	ϑ_7	ϑ_{15}	ϑ_{13}	ϑ_4	ϑ_6	ϑ_{14}	ϑ_{12}

The functions ϑ_{10}, ϑ_{11}, ϑ_5, ϑ_7, ϑ_{13}, ϑ_{14} are odd functions, the rest even.

55. For brevity denote

$$\begin{aligned} \vartheta(x+\xi, y+\eta) &\text{ by } \Theta, \\ \vartheta(x-\xi, y-\eta) &\text{ by } \Theta', \\ \vartheta(x, y) &\text{ by } \vartheta, \\ \vartheta(\xi, \eta) &\text{ by } \theta, \end{aligned}$$

for all the different suffixes. Also let

$$c_0=[\vartheta_0(x, y)]_{x=0, y=0}, \text{ \&c.}$$

for all the even functions.

Mr. FORSYTH finds formulæ for the products $\Theta\,\Theta'$ for different values of the suffixes. These are given on p. 834 *et seq.* of his Memoir. In the 13th set occur the first four of the following six formulæ. The remaining two formulæ are not in his list, but are proved in exactly the same way.

$$\begin{aligned}
c_4c_8\,\Theta_0\,\Theta'_{12}&=\theta_4\theta_8\,\vartheta_0\,\vartheta_{12}+\theta_1\,\theta_{13}\vartheta_5\vartheta_9-\theta_6\theta_{10}\vartheta_2\vartheta_{14}-\theta_7\,\theta_{11}\vartheta_3\,\vartheta_{15}\\
c_0c_2\,\Theta_{14}\Theta'_{12}&=\theta_0\theta_2\,\vartheta_{14}\vartheta_{12}+\theta_{14}\theta_{12}\vartheta_0\vartheta_2-\theta_5\theta_7\,\vartheta_9\vartheta_{11}-\theta_9\,\theta_{11}\vartheta_5\,\vartheta_7\\
c_2c_9\,\Theta_7\,\Theta'_{12}&=\theta_2\theta_9\,\vartheta_7\,\vartheta_{12}+\theta_7\,\theta_{12}\vartheta_2\vartheta_9+\theta_5\theta_{14}\vartheta_0\vartheta_{11}+\theta_0\,\theta_{11}\vartheta_5\,\vartheta_{14}\\
c_9c_{12}\Theta_9\,\Theta'_{12}&=\theta_9\theta_{12}\vartheta_9\,\vartheta_{12}+\theta_0\,\theta_5\,\vartheta_0\vartheta_5-\theta_2\theta_7\,\vartheta_2\vartheta_7-\theta_{11}\theta_{14}\vartheta_{11}\vartheta_{14}\\
c_2c_{12}\Theta_2\,\Theta'_{12}&=\theta_2\theta_{12}\vartheta_2\,\vartheta_{12}+\theta_0\,\theta_{14}\vartheta_0\vartheta_{14}+\theta_5\theta_{11}\vartheta_5\vartheta_{11}+\theta_9\,\theta_7\,\vartheta_9\,\vartheta_7\\
c_0c_9\,\Theta_5\,\Theta'_{12}&=\theta_0\theta_9\,\vartheta_5\,\vartheta_{12}+\theta_7\,\theta_{14}\vartheta_2\vartheta_{11}+\theta_5\theta_{12}\vartheta_0\vartheta_9+\theta_2\,\theta_{11}\vartheta_7\,\vartheta_{14}
\end{aligned}$$

If in each of these formulæ we change the sign of ξ, and then subtract each new formulæ from that from which it is derived, we deduce the following six equations :—

$$\begin{aligned}
c_4c_8\,(\Theta_0\,\Theta'_{12}-\Theta'_0\,\Theta_{12})&=2(\theta_1\,\theta_{13}\vartheta_5\vartheta_9-\theta_6\theta_{10}\vartheta_2\vartheta_{14})\\
c_0c_2\,(\Theta_{14}\Theta'_{12}-\Theta'_{14}\Theta_{12})&=2(\theta_{14}\theta_{12}\vartheta_0\vartheta_2-\theta_9\theta_{11}\vartheta_5\vartheta_7)\\
c_2c_9\,(\Theta_7\,\Theta'_{12}-\Theta'_7\,\Theta_{12})&=2(\theta_7\,\theta_{12}\vartheta_2\vartheta_9+\theta_0\theta_{11}\vartheta_5\vartheta_{14})\\
c_9c_{12}(\Theta_9\,\Theta'_{12}-\Theta'_9\,\Theta_{12})&=2(\theta_0\,\theta_5\,\vartheta_0\vartheta_5-\theta_2\theta_7\,\vartheta_2\vartheta_7)\\
c_2c_{12}(\Theta_2\,\Theta'_{12}-\Theta'_2\,\Theta_{12})&=2(\theta_0\,\theta_{14}\vartheta_0\vartheta_{14}+\theta_7\theta_9\,\vartheta_7\vartheta_9)\\
c_0c_9\,(\Theta_5\,\Theta'_{12}-\Theta'_5\,\Theta_{12})&=2(\theta_5\,\theta_{12}\vartheta_0\vartheta_9+\theta_2\theta_{11}\vartheta_7\vartheta_{14})
\end{aligned}$$

56. The odd functions $\vartheta_5, \ldots$ will vanish when $x=0$, $y=0$. Let c_5 be the coefficient of x in the expansion of ϑ_5, so that

$$c_5=\left\{\frac{d.\vartheta_5(x, y)}{dx}\right\}_{x=0, y=0}$$

and so on for all the odd functions.

Differentiate all the equations of the last set with respect to ξ and then put $\xi=0$. We notice that

$$\frac{d}{d\xi} f(x+\xi) = \frac{d}{dx} f(x+\xi)$$

and

$$\frac{d}{d\xi} f(x-\xi) = -\frac{d}{dx} f(x-\xi).$$

Hence, taking the first equation,

$$c_4 c_8 \left\{ \vartheta_{12} \frac{\vartheta d_0}{dx} - \vartheta_0 \frac{d\vartheta_{12}}{dx} \right\} = c_1 c_{13} \vartheta_5 \vartheta_9 - c_6 c_{10} \vartheta_2 \vartheta_{14}$$

or

$$c_4 c_8 \frac{d}{dx} \left(\frac{\vartheta_0}{\vartheta_{12}} \right) = c_1 c_{13} \frac{\vartheta_5}{\vartheta_{12}} \frac{\vartheta_9}{\vartheta_{12}} - c_6 c_{10} \frac{\vartheta_2}{\vartheta_{12}} \frac{\vartheta_{14}}{\vartheta_{12}}.$$

Performing the same operations on all the equations we arrive at the following six relations :—

$$c_4 c_8 \frac{d}{dx} \left(\frac{\vartheta_0}{\vartheta_{12}} \right) = c_1\, c_{13} \frac{\vartheta_5}{\vartheta_{12}} \frac{\vartheta_9}{\vartheta_{12}} - c_6 c_{10} \frac{\vartheta_2}{\vartheta_{12}} \frac{\vartheta_{14}}{\vartheta_{12}}.$$

$$c_0 c_2 \frac{d}{dx} \left(\frac{\vartheta_{14}}{\vartheta_{12}} \right) = c_{14} c_{12} \frac{\vartheta_0}{\vartheta_{12}} \frac{\vartheta_2}{\vartheta_{12}} - c_9 c_{11} \frac{\vartheta_5}{\vartheta_{12}} \frac{\vartheta_7}{\vartheta_{12}}.$$

$$c_2 c_9 \frac{d}{dx} \left(\frac{\vartheta_7}{\vartheta_{12}} \right) = c_7\, c_{12} \frac{\vartheta_2}{\vartheta_{12}} \frac{\vartheta_9}{\vartheta_{12}} + c_0 c_{11} \frac{\vartheta_5}{\vartheta_{12}} \frac{\vartheta_{14}}{\vartheta_{12}}.$$

$$c_9 c_{12} \frac{d}{dx} \left(\frac{\vartheta_9}{\vartheta_{12}} \right) = c_0\, c_5 \frac{\vartheta_0}{\vartheta_{12}} \frac{\vartheta_5}{\vartheta_{12}} - c_2 c_7 \frac{\vartheta_2}{\vartheta_{12}} \frac{\vartheta_7}{\vartheta_{12}}.$$

$$c_2 c_{12} \frac{d}{dx} \left(\frac{\vartheta_2}{\vartheta_{12}} \right) = c_0\, c_{14} \frac{\vartheta_0}{\vartheta_{12}} \frac{\vartheta_{14}}{\vartheta_{12}} + c_7 c_9 \frac{\vartheta_7}{\vartheta_{12}} \frac{\vartheta_9}{\vartheta_{12}}.$$

$$c_0 c_9 \frac{d}{dx} \left(\frac{\vartheta_5}{\vartheta_{12}} \right) = c_5\, c_{12} \frac{\vartheta_0}{\vartheta_{12}} \frac{\vartheta_9}{\vartheta_{12}} + c_2 c_{11} \frac{\vartheta_7}{\vartheta_{12}} \frac{\vartheta_{14}}{\vartheta_{12}}.$$

57. If we take the mass of the body to be unity, and notice that

$$A+F=B+G=C+H=1,$$

the equations of motion of the body under no forces may be written

$$\frac{A}{1-B-C} \dot{\omega}_1 + \omega_2 \omega_6 - \omega_3 \omega_5 = 0$$

$$\frac{B}{1-C-A} \dot{\omega}_2 + \omega_3 \omega_4 - \omega_1 \omega_6 = 0$$

$$\frac{C}{1-A-B} \dot{\omega}_3 + \omega_1 \omega_5 - \omega_2 \omega_4 = 0$$

$$\frac{1-A}{B-C} \dot{\omega}_4 + \omega_5 \omega_6 - \omega_2 \omega_3 = 0$$

$$\frac{1-B}{C-A} \dot{\omega}_5 + \omega_4 \omega_6 - \omega_1 \omega_3 = 0$$

$$\frac{1-C}{A-B} \dot{\omega}_6 + \omega_4 \omega_5 - \omega_1 \omega_2 = 0.$$

Assume, therefore,

$$\omega_1=a\frac{\vartheta_0(x,y)}{\vartheta_{12}(x,y)} \qquad \omega_4=f\frac{\vartheta_7(x,y)}{\vartheta_{12}(x,y)}$$

$$\omega_2=b\frac{\vartheta_2(x,y)}{\vartheta_{12}(x,y)} \qquad \omega_5=g\frac{\vartheta_5(x,y)}{\vartheta_{12}(x,y)}$$

$$\omega_3=c\frac{\vartheta_9(x,y)}{\vartheta_{12}(x,y)} \qquad \omega_6=h\frac{\vartheta_{14}(x,y)}{\vartheta_{12}(x,y)}$$

where in all these formulæ $x=nt+\alpha$ and y is arbitrary. The coefficients a, b, c, f, g, h, and n, α are arbitrary constants. If we substitute these values of $\omega_1, \omega_2, \ldots \omega_6$ in the equations of motion they give the equations

$$\frac{\mathrm{A}}{1-\mathrm{B}-\mathrm{C}}\,na\,\frac{d}{dx}\left(\frac{\vartheta_0}{\vartheta_{12}}\right)=cg\,\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}-bh\,\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}$$

$$\frac{\mathrm{B}}{1-\mathrm{C}-\mathrm{A}}\,nb\,\frac{d}{dx}\left(\frac{\vartheta_2}{\vartheta_{12}}\right)=ah\,\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}-cf\,\frac{\vartheta_9}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}$$

$$\frac{\mathrm{C}}{1-\mathrm{A}-\mathrm{B}}\,nc\,\frac{d}{dx}\left(\frac{\vartheta_9}{\vartheta_{12}}\right)=bf\,\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}-ag\,\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_5}{\vartheta_{12}}$$

$$\frac{1-\mathrm{A}}{\mathrm{B}-\mathrm{C}}\,nf\,\frac{d}{dx}\left(\frac{\vartheta_7}{\vartheta_{12}}\right)=bc\,\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_9}{\vartheta_{12}}-gh\,\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_{14}}{\vartheta_{12}}$$

$$\frac{1-\mathrm{B}}{\mathrm{C}-\mathrm{A}}\,ng\,\frac{d}{dx}\left(\frac{\vartheta_5}{\vartheta_{12}}\right)=ca\,\frac{\vartheta_9}{\vartheta_{12}}\frac{\vartheta_0}{\vartheta_{12}}-hf\,\frac{\vartheta_{14}}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}$$

$$\frac{1-\mathrm{C}}{\mathrm{A}-\mathrm{B}}\,nh\,\frac{d}{dx}\left(\frac{\vartheta_{14}}{\vartheta_{12}}\right)=ab\,\frac{\vartheta_0}{\vartheta_{12}}\frac{\vartheta_2}{\vartheta_{12}}-fg\,\frac{\vartheta_5}{\vartheta_{12}}\frac{\vartheta_7}{\vartheta_{12}}$$

Now these relations are of the same form as the relations between the ϑ's, which have been proved above. We have only to choose the arbitrary constants so as to make these equations exactly the same as the relations between the ϑ's, and then we have got a solution of the equations of motion.

58. Compare first the terms involving products; in this way we find the relations

$$\left.\begin{aligned} c_6c_{10}\,\frac{c}{h}&=\quad c_1c_{13}\,\frac{b}{g} \\ c_7c_9\,\frac{a}{f}&=-c_0c_{14}\,\frac{c}{h} \\ c_0c_5\,\frac{b}{g}&=\quad c_2c_7\,\frac{a}{f} \end{aligned}\right\}$$

Also

$$\left.\begin{aligned} c_0c_{11}\frac{b}{g}\frac{c}{h} &= -c_7c_{12} \\ c_2c_{11}\frac{a}{f}\frac{c}{h} &= -c_5c_{12} \\ c_9c_{11}\frac{a}{f}\frac{b}{g} &= c_{14}c_{12} \end{aligned}\right\}$$

If we eliminate a, b, c, f, g, h from the first three equations, by multiplication, the result takes the form

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0.$$

This equation among the constants does not hold in general; so we must suppose the parameters p, q, r, v, w of the theta-functions so connected as to make it satisfied.

Again, multiplying the last three equations by $\frac{a}{f}$, $\frac{b}{g}$, $\frac{c}{h}$, respectively, and comparing them, we see that

$$\frac{a}{f}\frac{c_7c_{12}}{c_0c_{11}}=\frac{b}{g}\frac{c_5c_{12}}{c_2c_{11}}=-\frac{c}{h}\frac{c_{12}c_{14}}{c_9c_{11}}.$$

Of these, the first equation is the same as the third of the first set.

The second may be written

$$\frac{b}{g}c_5c_9=-\frac{c}{h}c_2c_{14}$$

Comparing this with the first of the other set we get

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0$$

which is the same as before. Hence the six equations are consistent if this one relation holds.

They give us

$$\left(\frac{a}{f}\right)^2=-\frac{c_0^2c_5c_{14}}{c_2c_7^2c_9}\frac{b}{g}\frac{c}{h}$$

$$=\frac{c_0^2c_5c_{14}}{c_2c_7^2c_9}\frac{c_7c_{12}}{c_0c_{11}}$$

That is

$$\left(\frac{a}{f}\right)^2=\frac{c_0c_{12}c_5c_{14}}{c_2c_7c_9c_{11}}$$

Similarly

$$\left(\frac{b}{g}\right)^2=\frac{c_7c_{12}c_2c_{14}}{c_0c_9c_5c_{11}}$$

$$\left(\frac{c}{h}\right)^2=\frac{c_5c_7c_9c_{12}}{c_0c_2c_{11}c_{14}}.$$

59. Again, comparing the other terms of the equations with the relations between the θ's, we deduce six more relations

$$\frac{cg}{a}=\frac{c_1c_{13}}{c_4c_8}\frac{n\text{A}}{1-\text{B}-\text{C}}$$

$$\frac{ah}{b}=\frac{c_0c_{14}}{c_2c_{12}}\frac{n\text{B}}{1-\text{C}-\text{A}}$$

$$\frac{bf}{c}=-\frac{c_2c_7}{c_9c_{12}}\frac{n\text{C}}{1-\text{A}-\text{B}}$$

$$\frac{gh}{f}=\frac{c_0c_{11}}{c_2c_9}\frac{n(1-\text{A})}{\text{C}-\text{B}}$$

$$\frac{hf}{g}=\frac{c_2c_{11}}{c_0c_9}\frac{n(1-\text{B})}{\text{A}-\text{C}}$$

$$\frac{fg}{h}=\frac{c_9c_{11}}{c_0c_2}\frac{n(1-\text{C})}{\text{B}-\text{A}}$$

From the last three equations

$$f^2=-\frac{c_{11}^2}{c_0^2}n^2\frac{(1-\text{B})(1-\text{C})}{(\text{B}-\text{A})(\text{A}-\text{C})}$$

$$g^2=-\frac{c_{11}^2}{c_2^2}n^2\frac{(1-\text{C})(1-\text{A})}{(\text{C}-\text{B})(\text{B}-\text{A})}$$

$$h^2=-\frac{c_{11}^2}{c_9^2}n^2\frac{(1-\text{A})(1-\text{B})}{(\text{A}-\text{C})(\text{C}-\text{B})}.$$

These are all positive if A, C, B are in descending order of magnitude.

We have already found the ratios $\frac{a}{f}, \frac{b}{g}, \frac{c}{h}$ and therefore we know a, b, c, f, g, h completely in terms of A, B, C, and the c's. If we substitute these values in the first three equations of the last set, we are left with three relations among the constants. Now the Θ's implicitly involve the eight constants, p, q, r, v, w, n, α, and the other variable y, which is perfectly arbitrary. Among these quantities we have seen that there are four relations imposed by the form of the equations of motion. Hence there are four arbitrary constants left to express the initial conditions. But to do this completely, we should require six constants. Thus the solution is not complete.

60. The relation

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=0$$

may be simplified by means of general relations between the c's.

By Mr. FORSYTH's product theorem, we can prove that

$$-\theta_0\theta_{12}\vartheta_6\vartheta_{10}+\theta_3\theta_{15}\vartheta_5\vartheta_9+\theta_4\theta_8\vartheta_2\vartheta_{14}-\theta_7\theta_{11}\vartheta_1\vartheta_{13}=0$$

$$-\theta_0\theta_{12}\vartheta_5\vartheta_9+\theta_3\theta_{15}\vartheta_6\vartheta_{10}+\theta_4\theta_8\vartheta_1\vartheta_{13}-\theta_7\theta_{11}\vartheta_2\vartheta_{14}=0$$

$$\theta_0\theta_{12}\vartheta_2\vartheta_{14}+\theta_3\theta_{15}\vartheta_1\vartheta_{13}-\theta_4\theta_8\vartheta_6\vartheta_{10}-\theta_7\theta_{11}\vartheta_5\vartheta_9=0$$

$$\theta_0\theta_{12}\vartheta_1\vartheta_{13}+\theta_3\theta_{15}\vartheta_2\vartheta_{14}-\theta_4\theta_8\vartheta_5\vartheta_9-\theta_7\theta_{11}\vartheta_6\vartheta_{10}=0.$$

The last two equations are given by Mr. FORSYTH in equations (212), (219) of his Memoir. Differentiate these equations with regard to x and then put $x=0$, $y=0$, $\xi=0$, $\eta=0$; they give us

$$-c_0c_{12}\,.\,c_6c_{10}+c_3c_{15}\,.\,c_5c_9+c_4c_8\,.\,c_2c_{14}=0$$
$$-c_0c_{12}\,.\,c_5c_9+c_3c_{15}\,.\,c_6c_{10}+c_4c_8\,.\,c_1c_{13}=0$$
$$c_0c_{12}\,.\,c_2c_{14}+c_3c_{15}\,.\,c_1c_{13}-c_4c_8\,.\,c_6c_{10}=0$$
$$c_0c_{12}\,.\,c_1c_{13}+c_3c_{15}\,.\,c_2c_{14}-c_4c_8\,.\,c_5c_9=0.$$

Hence

$$\left.\begin{aligned}c_4c_8(c_2c_{14}-c_1c_{13})&=(c_0c_{12}+c_3c_{15})(c_6c_{10}-c_5c_9)\\ c_4c_8(c_6c_{10}-c_5c_9)&=(c_0c_{12}-c_3c_{15})(c_2c_{14}-c_1c_{13})\end{aligned}\right\}$$

Hence

$$c_4^2c_8^2=c_0^2c_{12}^2-c_3^2c_{15}^2$$

or

$$c_0^2c_{12}^2=c_4^2c_8^2+c_3^2c_{15}^2$$

But we have also

$$c_4c_8(c_2c_{14}+c_1c_{13})=(c_0c_{12}-c_3c_{15})(c_6c_{10}+c_5c_9)$$

Multiplying this by the first of the other two equations we get

$$c_2^2c_{14}^2-c_1^2c_{13}^2=c_6^2c_{10}^2-c_5^2c_9^2.$$

If therefore

$$c_2^2c_{14}^2c_1^2c_{13}^2=c_6^2c_{10}^2c_5^2c_9^2$$

we have also

$$c_2^2c_{14}^2+c_1^2c_{13}^2=c_6^2c_{10}^2+c_5^2c_9^2$$

whence

$$\left.\begin{aligned}c_6^2c_{10}^2&=c_2^2c_{14}^2\\ c_5^2c_9^2&=c_1^2c_{13}^2\end{aligned}\right\}$$

i.e.,

$$\left.\begin{aligned}c_6c_{10}&=\pm c_2c_{14}\\ c_5c_9&=\mp c_1c_{13}\end{aligned}\right\}$$

ADDITION.

(Added March 4, 1884.)

The relation between the constants may be put into another form. We have

$$c_0c_3\{\Theta_{15}\Theta'_{12}-\Theta_{12}\Theta'_{15}\}=2\{\theta_8\theta_{11}\vartheta_7\vartheta_4+\theta_5\theta_6\,\vartheta_9\vartheta_{10}\}$$
$$c_0c_{15}\{\Theta_3\,\Theta'_{12}-\Theta_{12}\Theta'_3\}=2\{\theta_4\theta_{11}\vartheta_7\vartheta_8-\theta_1\theta_{14}\vartheta_2\vartheta_{13}\}$$

Hence, performing the same operations as before

$$c_0c_3\frac{d}{dx}\left\{\frac{\vartheta_{15}}{\vartheta_{12}}\right\}=c_8c_{11}\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_4}{\vartheta_{12}}+c_5c_6\frac{\vartheta_9}{\vartheta_{12}}\frac{\vartheta_{10}}{\vartheta_{12}}$$

$$c_0c_{15}\frac{d}{dx}\left\{\frac{\vartheta_3}{d_{12}}\right\}=c_4c_{11}\frac{\vartheta_7}{\vartheta_{12}}\frac{\vartheta_8}{\vartheta_{12}}-c_1c_{14}\frac{\vartheta_2}{\vartheta_{12}}\frac{\vartheta_{13}}{\vartheta_{12}}.$$

Whence the coefficient of x in $\frac{d}{dx}\left(\frac{\vartheta_{15}}{\vartheta_{12}}\right)$

$$=\frac{c_8c_{11}c_7c_4+c_5c_6c_9c_{10}}{c_0c_3c_{12}^2}$$

and the coefficient of x in $\frac{d}{dx}\left(\frac{\vartheta_3}{\vartheta_{13}}\right)$

$$=\frac{c_4c_{11}c_7c_8-c_1c_{14}c_2c_{13}}{c_0c_{15}c_{12}^2}$$

Hence

$$c_5c_9c_6c_{10}+c_1c_{14}c_2c_{13}$$

$$=c_0c_{12}^2\times\text{ coefficient of } x \text{ in }\left\{c_3\frac{d}{dx}\left(\frac{\vartheta_{15}}{\vartheta_{12}}\right)-c_{15}\frac{d}{dx}\left(\frac{\vartheta_3}{\vartheta_{12}}\right)\right\}$$

$$=2c_0c_{12}^2\times\text{ coefficient of } x^2 \text{ in }\left\{\frac{c_3\vartheta_{15}-c_{15}\vartheta_3}{\vartheta_{12}}\right\}$$

Now

$$\vartheta_{15}=c_{15}-\tfrac{1}{2}x^2\left(\frac{\pi}{\mathrm{K}}\right)^2\frac{d}{dp'}c_{15}+\ldots$$

$$\vartheta_3=c_3-\tfrac{1}{2}x^2\left(\frac{\pi}{\mathrm{K}}\right)^2\frac{d}{dp'}c_3+\ldots$$

$$\vartheta_{12}=c_{12}-\tfrac{1}{2}x^2\left(\frac{\pi}{\mathrm{K}}\right)^2\frac{d}{dp'}c_{12}+\ldots$$

in Mr. FORSYTH's notation; therefore

$$c_3\vartheta_{15}-c_{15}\vartheta_3=\tfrac{1}{2}x^2\left(\frac{\pi}{\mathrm{K}}\right)^2\left\{c_{15}\frac{dc_3}{dp'}-c_3\frac{dc_{15}}{dp'}\right\},$$

and finally

$$c_6c_{10}c_5c_9+c_1c_{13}c_2c_{14}=c_0c_{12}\left(\frac{\pi}{\mathrm{K}}\right)^2\left\{c_{15}\frac{dc_3}{dp'}-c_3\frac{dc_{15}}{dp'}\right\}.$$

Hence, if this vanishes, we have

$$c_{15}\frac{dc_3}{dp}=c_3\frac{dc_{15}}{dp},\text{ since } p'=\log p;$$

that is, $\frac{c_3}{c_{15}}$ is independent of p.